Spectral Analysis of Random Closed Sets

The Surface Measure Associated with a Random Closed Set

Diplomarbeit

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Introduction

In many fields of natural sciences, medicine, and technology, there is need for simple but flexible models for random geometric structures and for appropriate statistical methods. Such structures appear, for example, in the field of forestry, such as positions of trees, in the field of geology, such as locations of ore deposits, or in the field of materials science, such as pores of a porous medium. The development and analysis of mathematical models used to describe these complicated random geometric patterns is the object of stochastic geometry. The most important models provided are random closed sets and random geometric fields.

A random closed set is formally defined as a set-valued random variable, that is, a random variable taking values on the space of closed subsets of $\mathbb{R}^d$ provided with a suitable topology. This concept, however, is often too general and difficult to trace mathematically. Therefore, only random closed sets are considered that have a representation as a union of a locally finite collection of compact convex sets, or convex bodies. Most real geometric structures to be analyzed can be approximated sufficiently well by such random closed sets.

A collection of closed sets is called a field of closed sets or a point process of closed sets. Point processes can be defined as counting measures on any measurable space. The best known case is that of $\mathbb{R}^d$, where the closed sets are the points of $\mathbb{R}^d$. Counting measures may also be interpreted as particular random measures. An introduction to the theory of random measures is given in [Kal83], and the connection with point processes is explained in [DVJ88].

In this thesis, we will mainly be concerned with random closed sets that have the important invariance property of stationarity. This property is defined as invariance of the distribution of the random set under translations by elements of $\mathbb{R}^d$. In materials science, samples are taken out of a material and analyzed. In order to be able to extend the results of a local analysis to the overall material, the material must be macroscopically homogeneous, that is, it must have the same properties at every location, so that where the sample is taken is of no importance. This macroscopical homogeneity corresponds to the concept of stationarity.

The term *Stochastic Geometry* was coined only in the 1970s and image analysis has been one of the driving forces in the rapid development of stochastic geometry over the past 30 years. Large data sets presented in the form of images need to be analyzed and processed automatically. Therefore, stochastic models are needed as descriptors of images as well as for their statistical analysis. One area that has had a major influence on stochastic geometry is integral geometry. G.Matheron used the various possibilities offered, for example, by the use of Minkowski measures to obtain indirect measurements of parameters that are difficult to measure otherwise. Minkowski, or curvature, measures are important quantities studied extensively in the context of integral geometry and stochastic geometry today.

Further areas that have influenced stochastic geometry are spatial statistics and probability theory. Many methods and techniques from these fields have been adapted to the problems arising in stochastic geometry. But although D.G.Kendall sees the beginning of stochastic geometry in the articles on spectral analysis of point processes by Bartlett, cf. [Bar63] and [Bar64], the theory of spectral analysis, a vital tool in signal and image processing, has not been integrated into the canon of stochastic geometry.

In signal and image processing, spectral analysis is used mainly to investigate spectral representations of second order quantities, such as covariograms and correlation functions. An image or a signal can be interpreted as a discrete function observed within a bounded window. Then the second order quantities describe the correlations between function values at two distinct points. The Fourier transformation is a powerful tool in this context and offers fast ways of calculation in applications.

In [Bar63], the spectral analysis of stationary point processes on the real line $\mathbb{R}$ is introduced. The results are then extended to two-dimensional point-processes in [Bar64]. The methods established by Bartlett are further applied to two-dimensional point processes in [RF83] and [MR96], and a way of interpreting the spectral quantity to gain information on the original process is explained for ecological data. In [DVJ88], the theory is extended from counting measures to stationary random measures, for which a second order quantity, the covariance measure, exists. The application of spectral methods to the analysis of random closed sets is proposed for the first time in [OM00].

The spectral analysis of the covariance function of a stationary random closed set is performed in [Koc02] and [KOS03]. The covariance function, which arises as the density of the reduced covariance measure associated with the volume measure of the random closed set, is a continuous, positive-definite function, so by an application of Bochner's theorem, the existence of an associated spectral quantity can be shown. The existence of this spectral quantity is then used to estimate the covariance function via frequency space.

In this thesis, a similar approach is used to establish the spectral analysis of second order quantities associated with the surface measure of a stationary random closed set. In contrast to the covariance function associated with the volume measure, the density of the surface covariance measure does not necessarily exist and even if it does, it is not continuous in general. Thus, difficulties arise which necessitate the use of different methods. We establish the existence of a spectral quantity associated with the surface covariance measure. Then the density of this measure is related to a tractable spectral quantity and the difficulties arising in the construction of a practicable estimator are discussed. Furthermore, a formula for the surface correlation function for a standard model of a random closed set is derived. Using the derivation process as a basis, a conjecture for a limit representation of the second moment measure is then formulated. Finally, the suggested estimator is applied to a standard model.

Chapter 1

Preliminaries

In this chapter we will give basic definitions and introduce the notations used throughout this work. Since the employed methodology stems mainly from the fields of stochastic geometry and Fourier analysis, we will concentrate on these. Standard results from measure and integration theory can be found, for example, in [Coh80] or [ADD00].

1.1 Stochastic Geometry and Integral Geometry

The following outline is based on [SW92], [SW00] and [MSSW90] and contains some results formulated by [Mat75].

1.1.1 General Notations

We work throughout in the Euclidean space $\mathbb{R}^d$. B^k denotes the k-dimensional unit ball, S^{k-1} the unit sphere in $\mathbb{R}^k$, and ν_k denotes k-dimensional Lebesgue measure on k-dimensional affine subspaces of $\mathbb{R}^d$. We put $\kappa_k := \nu_k(B^k)$ and $\omega_k = \nu_{k-1}(S^{k-1})$ in $\mathbb{R}^k$. In particular, we put $\nu_d =: \nu$. For $x, y \in \mathbb{R}^d$, $\langle x, y \rangle = \sum_{j=1}^{d} x_j y_j$ denotes the standard scalar product and $||x|| = \sqrt{\langle x, x \rangle}$ the Euclidean length of x.

For a subset $A \subset \mathbb{R}^d$ we use the notations ∂A and $\text{int}A$ to denote the boundary and the interior of A, respectively, and the notation $\overline{A}$ to denote the closure of A. Moreover, we write $\mathbb{1}_A$ for the characteristic or indicator function of the set A.

For subsets $A, B \subset \mathbb{R}^d$ and $c \in \mathbb{R}$, the *Minkowski sum* of A and B is defined as

$$A + B := \{a + b : a \in A, b \in B\}$$

and the multiplication by a scalar as

$$cA := \{ca : a \in A\}.$$

The Minkowski sum $A + \check{B}$ is called the *dilation* of A by B. We write $-A$ for $(-1)A$, $A - B$ for $A + (-B)$ and $A + x$ for $A + \{x\}$, where $x \in \mathbb{R}^d$. Furthermore, we write $\check{A} := -A$ for the *reflection* of the set A at the origin.

For a topological space $\mathbb{E}$, $\mathscr{B}(\mathbb{E})$ denotes the σ-algebra of Borel sets of $\mathbb{E}$. A *measure* μ is a non-negative real-valued set function defined on a σ-algebra $\mathcal{A}$ satisfying

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k), \quad A_k \in \mathcal{A}, A_i \cap A_j = \emptyset, i \neq j,$$

and

$$\mu(\emptyset) = 0.$$

If μ is allowed to take negative values, it is called a *signed measure*. A signed measure μ is called *locally finite* if $|\mu(A)| < \infty$ for any compact subset A of $\mathbb{R}^d$. We denote the space of all totally finite measures on $\mathbb{R}^d$ by $\mathscr{M}(\mathbb{R}^d)$ and the space of locally finite measures on $\mathbb{R}^d$ by $\mathscr{M}'(\mathbb{R}^d)$.

A *probability space* is a triple $(\Omega, \mathcal{A}, \mathbb{P})$, where Ω is a set, $\mathcal{A}$ is a σ-algebra on Ω, and $\mathbb{P}$ is a probability measure on $\mathcal{A}$, that is, a measure satisfying $\mathbb{P}(\Omega) = 1$.

1.1.2 Minkowski Functionals and Curvature Measures

A powerful tool for description and analysis of convex geometric objects is given by the Minkowski functionals W_j and the related curvature measures Φ_{d-j}. Let $\mathcal{K}$ denote the set of convex bodies in $\mathbb{R}^d$, that is, the set of nonvoid convex subsets of $\mathbb{R}^d$, and let $\mathcal{R}$ denote the set of all finite unions of convex bodies. $\mathcal{R}$ is called the convex ring. Furthermore, let $\mathscr{S}$ denote the extended convex ring, that is, the set of all subsets M of $\mathbb{R}^d$ such that $M \cap K$ is an element of $\mathcal{R}$ for each $K \in \mathcal{K}$.

We put $V_d(K) := \nu(K)$ for $K \in \mathcal{K}$, and call $V_d(K)$ the volume of K. For a convex body $K \in \mathcal{K}$ and $x \in \mathbb{R}^d$, $p(x, K)$ denotes the uniquely determined nearest point to x in K. It is called the *projection* of x onto K.

For the parallel body $K + \epsilon B^d$ of $K \in \mathcal{K}$ at distance $\epsilon \geq 0$, the *Steiner formula* holds, cf. Satz 2.2.1 in [SW92]. It expresses the volume of the parallel body as a polynomial in ϵ,

$$\begin{aligned} V_d(K + \epsilon B^d) &= \sum_{i=0}^{d} \epsilon^i \binom{d}{i} W_i(K) \\ &= \sum_{j=0}^{d} \epsilon^{d-j} \kappa_{d-j} V_j(K). \end{aligned}$$

In this way, the Minkowski functionals $W_0(= V_d), \ldots, W_d$ and the intrinsic volumes $V_0, \ldots, V_d$ are defined. These two series of functionals differ only in their respective normalization factors. More precisely, the relation

$$\kappa_{d-j} V_j(K) = \binom{d}{j} W_{d-j}(K)$$

holds for each $K \in \mathcal{K}$.

A local version of the Steiner formula, based on the *local* parallel set

$$U_\epsilon(K, A) = \{x \in \mathbb{R}^d : \|x - p(x, K)\| \le \epsilon, p(x, K) \in A\}$$

for a Borel subset $A \in \mathscr{B}(\mathbb{R}^d)$, leads to a series of Borel measures. Via

$$\mu_\epsilon(K, A) := \nu(U_\epsilon(K, A))$$

a finite Borel measure is defined for each $\epsilon > 0$. The local Steiner formula, cf. [SW92], states that, for each $A \in \mathscr{B}(\mathbb{R}^d)$ and each $\epsilon \ge 0$,

$$\mu_\epsilon(K, A) = \sum_{j=0}^{d} \epsilon^{d-j} \kappa_{d-j} \Phi_j(K, A) \tag{1.1}$$

with locally finite measures $\Phi_j(K, \cdot)$, cf. Satz 2.3.3 in [SW92].

The measure $\Phi_j(K, \cdot)$ is called the j-th *curvature measure* of the body $K \in \mathcal{K}$ for $j = 0, \ldots, d$. For $A = \mathbb{R}^d$ we have

$$\Phi_j(K, \mathbb{R}^d) = V_j(K) \quad \text{for } j = 0, \ldots, d,$$

that is, the total curvature measures coincide with the intrinsic volumes. Thus, the following properties of the curvature measures hold also for the intrinsic volumes.

For $j = 0, \ldots, d$ and $A \in \mathscr{B}(\mathbb{R}^d)$, the j-th curvature measure $\Phi_j(\cdot, A)$ is weakly continuous as a function of K, that is, $K_i \to K$ implies $\Phi_j(K_i, \cdot) \xrightarrow{w}$

$\Phi_j(K, \cdot)$ for $i \to \infty$. For each $A \in \mathscr{B}(\mathbb{R}^d)$, the function $\Phi_j(\cdot, A)$ is measurable on $\mathcal{K}$, and Φ_j is motion covariant, that is, $\Phi_j(gK, gA) = \Phi_j(K, A)$ for $K \in \mathcal{K}, A \in \mathscr{B}(\mathbb{R}^d)$ and any rigid motion g of $\mathbb{R}^d$. Moreover, Φ_j is j-th order homogeneous, that is, $\Phi_j(\alpha K, \alpha A) = \alpha^j \Phi_j(K, A)$ for $K \in \mathcal{K}, A \in \mathscr{B}(\mathbb{R}^d), \alpha > 0$, and Φ_j is defined locally, that is, for each open set $B \subset \mathbb{R}^d$ and all $K, M \in \mathcal{K}$ satisfying $K \cap B = M \cap B$ we have

$$\Phi_j(K, A) = \Phi_j(M, A)$$

for each Borel set $A \subset B$, cf. [SW92], Satz 2.3.5.

The curvature measures defined by (1.1) on the set $\mathcal{K}$ of convex bodies can be extended to the convex ring $\mathcal{R}$. There are several ways to define these extensions. Matheron showed that there exists a positive extension to the convex ring (cf. [Mat75]), but we will mainly be concerned with an additive extension, the existence of which was proved, for example, by Schneider and Weil in [SW92].

A functional $\varphi : \mathcal{K} \to X$ into a vector space X is called additive if $\varphi(\emptyset) = 0$ and

$$\varphi(K \cap M) + \varphi(K \cup M) = \varphi(K) + \varphi(M) \tag{1.2}$$

for all $K, M \in \mathcal{K}$ satisfying $K \cup M \in \mathcal{K}$. In the same way, a functional $\varphi : \mathcal{R} \to X$ is called additive if $\varphi(\emptyset) = 0$ and (1.2) holds for all $K, M \in \mathcal{R}$.

The values of the additive extension of Φ_j to the convex ring, which shall again be denoted by Φ_j, can be computed via the following equality, which is a straightforward extension of the inclusion-exclusion principle,

$$\Phi_j(K, A) = \sum_{v \in S(r)} (-1)^{|v|-1} \Phi_j(K_v, A),$$

where $K = \bigcup_{i=1}^{r} K_i, K_i \in \mathcal{K}$, $S(r)$ is the family of nonvoid subsets of $\{1, \dots, r\}$, $|v|$ is the number of elements of $v \in S(r)$, and for $v = \{i_1, \dots, i_k\}$, we put $K_v := K_{i_1} \cap \dots \cap K_{i_k}$.

The mappings $V_j : \mathcal{R} \to \mathbb{R}$ and $\Phi_j : \mathcal{R} \to \mathscr{M}'(\mathbb{R}^d)$ inherit the following properties from the respective mappings on $\mathcal{K}$. For $j = 0, \dots, d$, V_j is additive, j-th order homogeneous and motion invariant and Φ_j is additive, j-th order homogeneous and motion covariant, cf. Satz 2.4.4 in [SW92]. Furthermore, the extended curvature measures are measurable as functions of their first argument and they are defined locally, that is, for $K, M \in \mathcal{R}$ satisfying $K \cap G = M \cap G$ for some nonvoid open set $G \subset \mathbb{R}^d$, we have $\Phi_j(K, A) = \Phi_j(M, A)$ for each Borel set $A \subset G$.

For $j = 0, \dots, d-1$, the measure $\Phi_j(K, \cdot)$ is concentrated on the boundary ∂K of K. The d-th curvature measure of $K \in \mathcal{R}$ is just Lebesgue measure,

$$\Phi_d(K, A) = \nu(K \cap A) \quad \text{for } A \in \mathscr{B}(\mathbb{R}^d),$$

and for $d - 1$ the relation

$$2\Phi_{d-1}(K, A) = \mathscr{H}^{d-1}(\partial K \cap A)$$

holds for any $A \in \mathscr{B}(\mathbb{R}^d)$, provided that K is the closure of its interior. Here, $\mathscr{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. If $K \in \mathcal{K}$ and $\dim K = d - 1$, that is, if K is a subset of a hyperplane of $\mathbb{R}^d$, the factor 2 cancels. For the results stated here, cf. [SW92], p.55.

The curvature measures $\Phi_j(K, \cdot)$, $j = 0, \dots, d-1$, $K \in \mathcal{R}$, can also be introduced via the generalized curvature measures

$$\Theta_j(K, \cdot), \quad j = 0, \dots, d-1,$$

defined on $\mathscr{B}(\mathbb{R}^d \times S^{d-1})$, cf. [Sch93]. These measures contain an additional directional part of information and also satisfy a Steiner formula. The precise relation is

$$d\kappa_{d-j}\Phi_j(K, A) = \binom{d}{j}\Theta_j(K, A \times S^{d-1})$$

for bounded $A \in \mathscr{B}(\mathbb{R}^d)$. A detailed account of generalized curvature measures can be found in [Sch93].

1.1.3 Random Sets

Let $\mathcal{F}$ be the system of closed subsets of $\mathbb{R}^d$, $\mathcal{C}$ the system of compact subsets of $\mathbb{R}^d$ and $\mathcal{G}$ the system of open subsets of $\mathbb{R}^d$, each including the empty set $\emptyset$. For $A, A_1, \dots, A_k \subset \mathbb{R}^d$, put

$$\begin{aligned} \mathcal{F}^A &:= \{F \in \mathcal{F} : F \cap A = \emptyset\} \\ \mathcal{F}_A &:= \{F \in \mathcal{F} : F \cap A \neq \emptyset\} \end{aligned}$$

and

$$\mathcal{F}^A_{A_1,\dots,A_k} := \mathcal{F}^A \cap \mathcal{F}_{A_1} \cap \dots \cap \mathcal{F}_{A_k}, \quad k \in \mathbb{N}_0.$$

If $k = 0$, then $\mathcal{F}^A_{A_1,\dots,A_k} = \mathcal{F}^A$. The space $\mathcal{F}$ is topologized by the topology $\mathfrak{T}_f$ generated by the system

$$\{\mathcal{F}^C : C \in \mathcal{C}\} \cup \{\mathcal{F}_G : G \in \mathcal{G}\}.$$

The corresponding Borel σ-algebra $\mathscr{B}(\mathcal{F})$ is generated by each of the systems

$$\{\mathcal{F}_G : G \in \mathcal{G}\}, \{\mathcal{F}_C : C \in \mathcal{C}\}, \{\mathcal{F}^G : G \in \mathcal{G}\} \text{ and } \{\mathcal{F}^C : C \in \mathcal{C}\}.$$

By Satz 2.1.2 in [SW00], the space $\mathcal{F}$ is compact with countable base.

We are now in a position to define random closed sets. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A *random closed set* (RACS) is a Borel measurable mapping $\Xi : \Omega \to \mathcal{F}$. The *distribution* of Ξ is the image measure $\mathbb{P}_\Xi$ under Ξ of $\mathbb{P}$. Two RACS Ξ, Ξ' having the same distribution are called *(stochastically) equivalent* ($\Xi \sim \Xi'$).

A RACS Ξ is called *stationary* if its distribution is invariant under translations in $\mathbb{R}^d$, that is, if $\Xi_x \sim \Xi$ for all $x \in \mathbb{R}^d$, and it is called *isotropic* if its distribution is invariant under rotations in $\mathbb{R}^d$, that is, if $\vartheta\Xi \sim \Xi$ for each rotation $\vartheta \in SO(\mathbb{R}^d)$. We will be concerned mainly with stationary RACS, and occasionally assume isotropy.

Characteristics of a Stationary RACS Ξ

In analogy to the distribution function of a random variable, one can define, for each RACS Ξ, a functional

$$\begin{aligned} T : \mathcal{C} &\to [0,1] \\ C &\mapsto \mathbb{P}_\Xi(\mathcal{F}_C) = \mathbb{P}(\Xi \cap C \neq \emptyset), \end{aligned}$$

which can be characterized as an alternating Choquet capacity of infinite order. The functional T is called the *capacity functional* of the RACS Ξ in $\mathbb{R}^d$. It has the following properties:

1. $0 \leq T \leq 1, T(\emptyset) = 0,$
2. $T(C_i) \searrow T(C)$ for $C_i, C \in \mathcal{C}, C_i \searrow C,$
3. $S_k(C; C_1, \ldots, C_k) \geq 0$ for $C, C_1, \ldots, C_k \in \mathcal{C}, k = 0, 1, 2, \ldots,$ where

   $$S_0(C) := 1 - T(C)$$

   and

   $$S_k(C; C_1, \ldots, C_k) := S_{k-1}(C; C_1, \ldots, C_{k-1}) - S_{k-1}(C \cup C_k; C_1, \ldots, C_{k-1})$$

   for $k = 1, 2, \ldots$

Conversely, there exists, for each functional $T : \mathcal{C} \to [0,1]$ satisfying the three conditions above, a unique (up to equivalence) RACS Ξ whose associated functional is T.

For the RACS Ξ one can define a mean function p and a *covariance function* cov_V via the corresponding stochastic field $\mathbf{1}_\Xi$,

$$p(x) = \mathbb{E}\mathbf{1}_\Xi(x), \quad x \in \mathbb{R}^d,$$

$$\mathrm{cov}_V(x,y) = \mathbb{E}(\mathbf{1}_\Xi(x) - p(x))(\mathbf{1}_\Xi(y) - p(y)), \quad x, y \in \mathbb{R}^d.$$

In the stationary case, we have

$$p(x) = p(0) =: p$$

and

$$\mathrm{cov}_V(x,y) = \mathrm{cov}_V(x-y, 0) =: \mathrm{cov}_{V,0}(x).$$

For the constant p,

$$p = \mathbb{P}(o \in \Xi) = \frac{\mathbb{E}\nu(\Xi \cap A)}{\nu(A)}$$

for each Borel set $A \subset \mathbb{R}^d$ satisfying $\nu(A) > 0$. For that reason, p is called the *volume fraction* or the *volume density* of Ξ.

For the covariance function we have

$$\mathrm{cov}_{V,0}(x) = \mathbb{P}(o \in \Xi, x \in \Xi) - p^2.$$

$C_V(x) = \mathrm{cov}_{V,0}(x) + p^2$ is called the *covariance* of Ξ. Since

$$C_V(x) = \mathbb{P}(o \in \Xi \cap (\Xi - x)),$$

$C_V(x)$ is the volume fraction of the RACS $\Xi \cap (\Xi - x)$. If Ξ is isotropic, $C_V(x)$ depends solely on the length $\|x\|$ of x. Furthermore, for a stationary RACS Ξ we have

$$p = \mathbb{P}(o \in \Xi) = T_\Xi(\{0\}) = \mathbb{E}\nu(\Xi \cap C^d),$$

where C^d denotes the d-dimensional unit cube.

The measure $V_\Xi(\Xi \cap A) := \nu(\Xi \cap A)$ is called the *volume measure* of the RACS Ξ. We use the subscript V in this context to denote the relation of the covariance function to the volume measure.

1.1.4 Point Processes

Point processes can be interpreted as particular RACS but also have a representation as random counting measures. The use of point processes allows the definition of an important class of RACS called the *Boolean model* which is both flexible and quite well tractable mathematically. This model is introduced at the end of the section. The following outline is based on [SW00].

A point $x \in \mathbb{R}^d$ can be identified with the point measure δ_x. This measure is defined by

$$\delta_x(A) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases}$$

for $A \in \mathscr{B}(\mathbb{R}^d)$. It is a probability measure on $\mathscr{B}(\mathbb{R}^d)$. A finite or countable sum

$$\eta := \sum_{i=1}^{k} \delta_{x_i}, \quad k \in \mathbb{N} \cup \{\infty\},$$

defines a measure on $\mathscr{B}(\mathbb{R}^d)$; it is called a *counting measure*. If $\eta(\{x\}) \leq 1$ holds for all $x \in \mathbb{R}^d$, then the counting measure is called *simple*.

Let $\mathscr{N}' = \mathscr{N}'(\mathbb{R}^d)$ be the set of all locally finite counting measures on $\mathbb{R}^d$ and $\mathscr{N}'_e$ the set of simple measures in $\mathscr{N}'$. Furthermore, let $\mathcal{N}$ denote the σ-algebra generated by the mappings

$$\begin{aligned} \Phi_A : \mathscr{N}' &\rightarrow \mathbb{R} \cup \{\infty\} \\ \eta &\mapsto \eta(A) \end{aligned}$$

for $A \in \mathscr{B}(\mathbb{R}^d)$. The support of η is defined by

$$\operatorname{supp} \eta := \{x \in \mathbb{R}^d : \eta(\{x\}) \geq 1\}.$$

It is a locally finite and closed subset of $\mathbb{R}^d$.

A *point process* in $\mathbb{R}^d$ is a measurable mapping Ψ from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to the measurable space $(\mathscr{N}', \mathcal{N})$. The distribution of the point process Ψ is the image measure $\mathbb{P}_\Psi$. A point process Ψ is called *simple* if $\Psi \in \mathscr{N}'_e$ a.s. By Satz 3.1.2 in [SW00], a simple point process Ψ is isomorphic to the locally finite RACS supp Ψ. Hence, a point process can be interpreted either as a counting measure or as a locally finite set. Stationarity and isotropy of a point process are defined analogously as for a RACS via translation and rotation invariance of its distribution.

The *intensity measure* of a point process Ψ, defined by

$$\Lambda(A) := \mathbb{E}\Psi(A) \quad \text{for } A \in \mathscr{B}(\mathbb{R}^d),$$

corresponds to the expectation of a random variable. If Ψ is simple, then $\Lambda(A)$ gives the mean number of points of Ψ lying in A. Although $\Lambda(C)$ can be infinite for some $C \in \mathcal{C}$, we will always assume Λ to be locally finite in the following. If Ψ is stationary, then the intensity measure is given by

$$\Lambda(A) = \lambda\nu(A),$$

with a constant $\lambda \in [0, \infty)$, which is called the *intensity* of the process.

The most important class of point processes are Poisson point processes. A simple point process Ψ with intensity measure Λ is called a *Poisson process*, if for all $A \in \mathscr{B}(\mathbb{R}^d)$ with $\Lambda(A) < \infty$, the a.s. real-valued random variable $\Psi(A)$ has a Poisson distribution.

The definition of a point process stated above has been formulated in the locally compact space $\mathbb{R}^d$. More generally, point processes can be defined on any locally compact space E with countable base, such as $E = \mathbb{R}^d \times M$, where M is a locally compact space with countable base and E is endowed with the product topology. If a point process Ψ on $\mathbb{R}^d \times M$ satisfies

$$\Lambda(C \times M) < \infty \quad \text{for all } C \in \mathcal{C},$$

it is called a *marked point process* in $\mathbb{R}^d$ with mark space M. A pair (y, m) is then interpreted as a point $y \in \mathbb{R}^d$ endowed with a mark $m \in M$.

The translation by $x \in \mathbb{R}^d$ on $\mathbb{R}^d \times M$ operates on $\mathbb{R}^d$ only, so that stationarity of the marked point process is understood with respect to translations of its first component. For a stationary marked point process Ψ in $\mathbb{R}^d$ with mark space M and $\Lambda \not\equiv 0$, we have

$$\Lambda = \lambda\nu \otimes \mathbb{Q}$$

with $0 < \lambda < \infty$ and a unique probability measure $\mathbb{Q}$ on M, cf. Satz 3.4.1 in [SW00]. λ is again called the intensity of the marked point process Ψ and the measure $\mathbb{Q}$ is called the *mark distribution* of Ψ.

A marked point process Ψ on $E = \mathbb{R}^d \times \mathcal{C}'$, where $\mathcal{C}' := \mathcal{C}\backslash\{\emptyset\}$, is called a *germ-grain process*. Here, the points x of the pairs (x, C) are interpreted as germs and the compact sets $C + x$ as grains. Given a germ-grain process Ψ, a RACS Ξ_Ψ can be constructed as the union set

$$\Xi_\Psi := \bigcup_{(x,C)\in \text{ supp } \Psi} (C + x).$$

Ξ_Ψ is called a *germ-grain model.* Conversely, every RACS Ξ can be represented as the union set of a germ-grain process Ψ such that Ψ inherits the invariance properties of Ξ, cf. Satz 4.4.2 in [SW00].

Another possible example for the space E is the space $\mathcal{E}_k^d$, $k = 1, \ldots, d-1$, of k-dimensional affine subspaces of $\mathbb{R}^d$. A point process on $\mathcal{E}_k^d$ is called a *k-plane process.* If $k = d-1$, the process is called a *hyperplane process.* Let $\mathcal{L}_k^d$ denote the set of k-dimensional linear subspaces of $\mathbb{R}^d$ and for $L \in \mathcal{L}_k^d$ let $L^\perp$ denote the orthogonal space of L. The intensity measure Λ of a stationary k-plane process Ψ can be written, for all $A \in \mathscr{B}(\mathcal{E}_k^d)$, as

$$\Lambda(A) = \lambda \int_{\mathcal{L}_k^d} \int_{L^\perp} \mathbb{1}_A(L + x)\nu_{d-k}(\mathrm{d}x)\mathbb{P}_0(\mathrm{d}L)$$

with $0 < \lambda < \infty$ and a probability measure $\mathbb{P}_0$ on $\mathcal{L}_k^d$, cf. Korollar 4.1.2 in [SW00]. λ is called the intensity and $\mathbb{P}_0$ the *directional distribution* of Ψ. λ and $\mathbb{P}_0$ are uniquely determined by Λ. If Ψ is also isotropic, then $\mathbb{P}_0$ is invariant under rotations.

For a stationary hyperplane process Ψ with intensity λ,

$$\mathbb{E} \sum_{E\in\Psi} \nu_{d-1}(E \cap A) = \lambda\nu(A)$$

for $A \in \mathscr{B}(\mathbb{R}^d)$, cf. Satz 4.1.4 in [SW00].

In the point process context, the following theorem is of great importance, comparable to Fubini's theorem of integration theory. A proof can be found, for example, in [SW00], Satz 3.1.5.

Theorem 1.1.1 (Campbell) Let Ψ be a point process in $\mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ a nonnegative measurable function. Then $\sum_{x\in\mathbb{R}^d} \Psi(\{x\})f(x)$ is measurable and the following equalities hold,

$$\mathbb{E} \sum_{x\in\mathbb{R}^d} \Psi(\{x\})f(x) = \mathbb{E} \int_{\mathbb{R}^d} f(x)\Psi(\mathsf{d}x) = \int_{\mathbb{R}^d} f(x)\Lambda(\mathsf{d}x).$$

Since Ψ is interpreted as a random counting measure here, the Campbell theorem can be generalized to random measures on $\mathbb{R}^d$, cf. [DVJ88], p.188. A random measure μ is a measurable mapping from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to the measurable space $(\mathscr{M}', \mathcal{M})$, where the σ-algebra $\mathcal{M}$ is defined analogously as $\mathcal{N}$. If μ is a random measure on $\mathbb{R}^d$ with expectation measure $M = \mathbb{E}\mu$, then the Campbell theorem for random measures yields the equality

$$\mathbb{E} \int_{\mathbb{R}^d} f(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}^d} f(x)M(\mathrm{d}x)$$

for any nonnegative measurable function $f : \mathbb{R}^d \to \mathbb{R}$.

Boolean Models

We can now introduce the most important model of RACS in applications, that is, the Boolean model. Boolean models can be used both to describe a particular empirical random set and to approximate more complicated random sets. An introduction to the topic can be found in [SKM95].

Let Ψ with supp $\Psi = \{x_1, x_2, \ldots\}$ be a stationary Poisson process in $\mathbb{R}^d$ with intensity λ and let $\Xi_1, \Xi_2, \ldots$ be a sequence of independent, identically distributed random compact sets in $\mathbb{R}^d$ that are independent of Ψ and such that

$$\mathbb{E}\nu(\Xi_0 \cap C) < \infty \quad \text{for all } C \in \mathcal{C}.$$

Here, Ξ_0 denotes a further random compact set of the same distribution as the Ξ_i but independent both of them and of Ψ. The *Boolean model with primary grain* Ξ_0 is the stationary germ-grain model $\Xi_{\tilde{\Psi}}$ obtained as the union set of the marked point process $\tilde{\Psi}$ with supp $\tilde{\Psi} = \{(x_1, \Xi_1), (x_2, \Xi_2), \ldots\}$,

$$\Xi_{\tilde{\Psi}} = \bigcup_{(x_i,\Xi_i)\in \text{ supp } \tilde{\Psi}} (\Xi_i + x_i).$$

The Boolean model $\Xi_{\tilde{\Psi}}$ is uniquely determined (up to stochastic equivalence) by the intensity measure of the Poisson process of germs and the distribution $\mathbb{Q}$ of the primary grain Ξ_0, cf. [SW00], p.150. Since the properties of the underlying point process are not explicitly made use of in this work, we will omit the subscript Ψ and use Ξ to denote Boolean models as well as general RACS.

The volume fraction and covariance of a stationary Boolean model Ξ are given by

$$p = 1 - \exp(-\lambda \mathbb{E}\nu(\Xi_0))$$

and

$$C_V(x) = 2p - 1 + (1-p)^2 \exp(\lambda \mathbb{E}\nu(\Xi_0 \cap (\Xi_0 - x))), \quad x \in \mathbb{R}^d,$$

respectively, cf. sect.3.1.1 in [SKM95].

1.2 Fourier Analysis

1.2.1 General Notation

In the following, let $f : \mathbb{R}^d \to \mathbb{C}$. A function f is called *measurable* if it is measurable with respect to the σ-algebra $\mathscr{B}(\mathbb{R}^d)$ of Borel subsets of $\mathbb{R}^d$. It is then also Lebesgue measurable. Furthermore, a measurable function is *integrable*, if it is integrable with respect to Lebesgue measure. A function f is called *essentially bounded* if it is bounded outside a set of Lebesgue measure zero.

For any $p \in [1, +\infty)$, the space of all measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ for which

$$\int |f(x)|^p \mathrm{d}x < \infty,$$

is denoted by $\mathcal{L}^p(\mathbb{R}^d)$. With $N_p := \{f : f = 0 \text{ a.e.}\}$, the quotient vector space

$$L^p(\mathbb{R}^d) := \mathcal{L}^p(\mathbb{R}^d)/N_p$$

is obtained. It consists of equivalence classes of functions that coincide almost everywhere (with respect to Lebesgue measure), but usually the elements of $L^p(\mathbb{R}^d)$ are treated as functions. The space $L^p(\mathbb{R}^d)$ is endowed with a norm via

$$f \mapsto \|f\|_{L^p} := \Big(\int_{\mathbb{R}^d} |f(x)|^p \mathrm{d}x \Big)^{\frac{1}{p}}.$$

Similarly, on the space $L^\infty(\mathbb{R}^d)$ of all equivalence classes of essentially bounded functions $f : \mathbb{R}^d \to \mathbb{C}$ a norm is defined by

$$f \mapsto \|f\|_{L^\infty} := \inf_{\substack{A \in \mathscr{B}(\mathbb{R}^d) \\ \nu(A)=0}} \sup_{x \in \mathbb{R}^d \setminus A} |f(x)|.$$

With the norms given above, the spaces $(L^p(\mathbb{R}^d), \|\cdot\|_{L^p})$ are complex Banach spaces for each $p \in [1, \infty]$.

A further space that is of great importance in the context of Fourier analysis is the Schwartz space $\mathscr{S}(\mathbb{R}^d)$ of infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{C}$ that, together with all their derivatives, converge to zero at infinity faster than any inverse power of x. We have $\mathscr{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ for any $p \in [1, \infty)$, cf. Satz 2.1 in part II of [SD80].

The *convolution* of two measurable functions $f, g : \mathbb{R}^d \to \mathbb{C}$ is the mapping defined by

$$\begin{aligned} f * g : \mathbb{R}^d &\to \mathbb{R} \cup \{-\infty, \infty\} \\ x &\mapsto \int_{\mathbb{R}^d} f(y)\overline{g(x-y)}\mathrm{d}y, \end{aligned}$$

where the bar denotes complex conjugation. We use the notations

$$f'(x) := f(-x), \quad x \in \mathbb{R}^d,$$

to denote reflection of f at the origin and

$$f^*(x) := \overline{f'(x)}, \quad x \in \mathbb{R}^d,$$

to denote involution so that

$$(f * g^*)(x) = \int_{\mathbb{R}^d} f(y)g(y-x)\mathrm{d}y.$$

Similar to the convolution of functions, the convolution of a measure μ on $\mathscr{B}(\mathbb{R}^d)$ and a function $f : \mathbb{R}^d \to \mathbb{R}$, which is measurable with respect to μ, is defined by

$$\begin{aligned} f * \mu : \mathbb{R}^d &\to \mathbb{R} \cup \{-\infty, \infty\} \\ x &\mapsto \int_{\mathbb{R}^d} f(x-y)\mu(\mathrm{d}y). \end{aligned}$$

1.2.2 Fourier Transformation

The *Fourier transform* of a function $f \in L^1(\mathbb{R}^d)$ is defined by

$$\tilde{f} := \mathscr{F}f : \mathbb{R}^d \ni \xi \mapsto \frac{1}{\sqrt{2\pi}^d} \int_{\mathbb{R}^d} f(x)e^{-i\langle \xi, x\rangle}\mathrm{d}x \in \mathbb{C}.$$

Analogously, the *Fourier cotransform* of f is defined by

$$\overline{\mathscr{F}}f : \mathbb{R}^d \ni x \mapsto \frac{1}{\sqrt{2\pi}^d} \int_{\mathbb{R}^d} f(\xi) e^{i\langle x,\xi\rangle} \mathrm{d}\xi \in \mathbb{C}.$$

By the Riemann-Lebesgue theorem, cf. [SD80], part II, Satz 1.1, $\tilde{f}(\xi) \to 0$ as $\|\xi\| \to \infty$, that is, the Fourier transform maps the space $L^1(\mathbb{R}^d)$ into the space $C_0(\mathbb{R}^d)$ of continuous functions that vanish at infinity. The Fourier transform and cotransform are linear. For a function $f \in L^1(\mathbb{R}^d)$ for which $\tilde{f} \in L^1(\mathbb{R}^d)$, the inversion formula

$$\overline{\mathscr{F}} \circ \mathscr{F} f = f, \quad \mathscr{F} \circ \overline{\mathscr{F}} f = f,$$

holds, cf. [SD80], part II, Satz 1.8. For the Schwartz space $\mathscr{S}(\mathbb{R}^d)$, we have

$$\mathscr{F}(\mathscr{S}(\mathbb{R}^d)) \to \mathscr{S}(\mathbb{R}^d) \quad \subset L^2(\mathbb{R}^d).$$

As a consequence of the Parseval identity

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}\mathrm{d}x = \int_{\mathbb{R}^d} \tilde{f}(\xi)\overline{\tilde{g}(\xi)}\mathrm{d}\xi$$

and the Plancherel identity

$$\int_{\mathbb{R}^d} \left|f(x)\right|^2 \mathrm{d}x = \int_{\mathbb{R}^d} \left|\tilde{f}(\xi)\right|^2 \mathrm{d}\xi$$

for $f, g \in \mathscr{S}(\mathbb{R}^d)$, cf. [SD80], part II, Satz 2.4, the Fourier transform and the Fourier cotransform can be uniquely extended to mutually inverse isomorphisms on the Hilbert space $L^2(\mathbb{R}^d)$.

In the literature, different definitions of the Fourier transform are used. Sometimes the factor $\frac{1}{\sqrt{2\pi}^d}$ is omitted for the Fourier transform and the cotransform is normalized by $\frac{1}{(2\pi)^d}$. Also, in the exponents, i and $-i$ are sometimes switched or replaced by $\pm 2\pi i$, where in the latter case the factor $\frac{1}{\sqrt{2\pi}^d}$ disappears.

1.2.3 Bochner's Theorem

A very important theorem needed in this work is Bochner's theorem. It provides the necessary means to establish the existence of a spectral quantity. In order to be able to use it, we need to introduce the notion of positive-definiteness for functions on $\mathbb{R}^d$. An introduction to the theory of positive-definite functions on groups is given in [Sas94]. Since we will only consider functions on $\mathbb{R}^d$, we adapt the formulations given there appropriately.

Definition 1.2.1 A function $f : \mathbb{R}^d \to \mathbb{C}$ is called *positive-definite*, if for all finite systems $\{x_1, \ldots, x_n\} \subset \mathbb{R}^d$ and $\{c_1, \ldots, c_n\} \subset \mathbb{C}$, the following inequality holds

$$\sum_{i,j=1}^{n} f(x_i - x_j)c_i\overline{c_j} \geq 0.$$

If two functions $f, g : \mathbb{R}^d \to \mathbb{C}$ are positive-definite, then so are the functions $\overline{f}, f^*, \mathrm{Re}(f), |f|^2$, and fg. Moreover, for all $a, b \geq 0$, the function $af + bg$ is positive-definite, cf. [Sas94], Thm.1.3.2.

If $f : \mathbb{R}^d \to \mathbb{C}$ is continuous, then the above condition is equivalent to

$$\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \varphi(x)\overline{\varphi(y)}f(x-y)\mathrm{d}x\mathrm{d}y \geq 0$$

for $\varphi \in L^1(\mathbb{R}^d)$ or $\varphi \in C_c(\mathbb{R}^d)$, the continuous functions of compact support, cf. [Boc32].

The notion of positive-definiteness arises naturally in the context of Fourier analysis and it has proved particularly valuable in the development of harmonic analysis on groups. The space $\mathbb{R}^d$ can be interpreted as a locally compact group. Positive-definiteness is defined analogously as stated above for any group G, where in this case the group operation is written additively. It can be shown that the continuous positive-definite functions on the group G are closely connected with the unitary representations of G, cf. [Sas94], sect. 1.3. This connection has been used to develop a harmonic analysis for unitary representations in terms of integral decompositions of the associated positive-definite functions.

A particularly important result in this context is Bochner's theorem, which we can now formulate.

Theorem 1.2.1 (Bochner) Let $f : \mathbb{R}^d \to \mathbb{C}$ be a continuous, positive-definite function. Then there exists a unique positive Radon measure μ with finite total mass $\mu(\mathbb{R}^d) = f(0)$ and

$$f(x) = \frac{1}{\sqrt{2\pi}^d} \int\limits_{\mathbb{R}^d} e^{i\langle\xi,x\rangle}\mu(\mathsf{d}\xi), \quad x \in \mathbb{R}^d.$$

Conversely, for any positive measure $\mu \in \mathscr{M}(\mathbb{R}^d)$, the Fourier-Stieltjes cotransform

$$\overline{\mathscr{F}}\mu(x) = \frac{1}{\sqrt{2\pi}^d} \int\limits_{\mathbb{R}^d} e^{i\langle\xi,x\rangle}\mu(\mathsf{d}\xi), \quad x \in \mathbb{R}^d,$$

is a positive-definite function on $\mathbb{R}^d$ satisfying $\mu(\mathbb{R}^d) = \overline{\mathscr{F}}\mu(0)$.

A Radon measure on the locally compact space $\mathbb{R}^d$ is a signed measure taking finite values on compact sets and having further properties which will not be needed in this work. Details can be found, for example, in [Coh80]. The proof of the first part of the theorem is given in [SD80], part II, Satz 3.7. For the second part, see [SD80], part II, Satz 3.2.

Chapter 2

Spectral Analysis of Stationary Random Closed Sets

In this chapter, spectral methods are used to analyze the second order surface structure of a stationary random closed set. Since the probability of a point lying on the surface of a stationary RACS is zero, it is impossible to define second order probability functions for the surface directly. Therefore the surface measure of the random closed set is considered. Quantities related to the second order surface structure then arise as the densities of the associated reduced second moment and reduced covariance measure.
In the first section, the surface measure is defined and important properties are introduced. Then the existence of a spectral measure associated with the reduced covariance measure is proved. The second section contains a proof of an analogon of the theorem of Wiener-Khintchine, which is of basic importance in signal analysis. Based on this result, the construction of an estimator for the surface correlation function is discussed, which concludes this chapter.

2.1 The Surface Measure of a Stationary Random Closed Set

2.1.1 Definition and Properties

In the following, let Ξ be a stationary random closed set taking values a.s. in the extended convex ring $\mathcal{S}$. This assumption allows the use of the curvature measures defined in the first chapter for statistical analysis of Ξ. Such a RACS is called a random $\mathcal{S}$-set.

For a deterministic set K in the extended convex ring that is the closure of its interior, the measure $2\Phi_{d-1}(K,\cdot)$ measures the surface content of K within A for any bounded Borel set A, that is,

$$2\Phi_{d-1}(K,A) = \mathscr{H}^{d-1}(A \cap \partial K).$$

For this reason and since, for any bounded $A \in \mathscr{B}(\mathbb{R}^d)$, $\Phi_{d-1}(\cdot, A)$ is locally defined, we can use it to define the surface measure of the random closed set Ξ, provided Ξ is a.s. the closure of its interior. If $A \in \mathscr{B}(\mathbb{R}^d)$ is bounded and $L \in \mathcal{K}$ is such that $A \subset \operatorname{int} L$, then

$$\Phi_{d-1}(\Xi, A) := \Phi_{d-1}(\Xi \cap L, A)$$

is well-defined, measurable, and nonnegative. That is, $\Phi_{d-1}(\Xi, A)$ is a random variable for any bounded Borel set A and thus, by Prop.6.1.III in [DVJ88], the mapping

$$\begin{aligned} S_\Xi : \mathscr{B}(\mathbb{R}^d) &\rightarrow [0,\infty) \\ A &\mapsto 2\Phi_{d-1}(\Xi, A) \end{aligned}$$

defines a locally finite random measure on $\mathscr{B}(\mathbb{R}^d)$. S_Ξ is called the *surface measure* of the stationary RACS Ξ. Its *distribution* is the probability measure it induces on the space $\mathscr{M}'(\mathbb{R}^d)$ of locally finite measures on $\mathbb{R}^d$.

If Ξ is a locally finite union of subsets of hyperplanes of $\mathbb{R}^d$, for example, if Ξ is the union set of a hyperplane process, then the equality $\Phi_{d-1}(\Xi, A) = \mathscr{H}^{d-1}(\partial \Xi \cap A)$ holds for any bounded Borel set A, so that in this case the factor 2 in the definition of the surface measure must be omitted.

It is clear that S_Ξ inherits stationarity from Ξ, which means that the distribution $\mathbb{P}_{S_\Xi}$ of S_Ξ is invariant under the shifts

$$\hat{T}_u \mathbb{P}(B) = \mathbb{P}(T_u B)$$

for $B \in \mathscr{B}(\mathscr{M}'(\mathbb{R}^d))$, where $T_u B = \{T_u \mu : \mu \in B\}$ and $T_u\mu$ is the shift operator induced on $\mathscr{M}'(\mathbb{R}^d)$ by the translations on $\mathbb{R}^d$

$$(T_u\mu)(A) = \mu(A+u), \quad A \in \mathscr{B}(\mathbb{R}^d),$$

cf. [DVJ88], ch. 10.1.

In analogy to the intensity measure of a point process, the *first moment measure* of S_Ξ is defined by

$$M_{\Xi,S}(A) := \mathbb{E} S_\Xi(A)$$

for bounded $A \in \mathscr{B}(\mathbb{R}^d)$. By stationarity of S_Ξ, $M_{\Xi,S}$ is translation invariant. It coincides, up to a constant factor, with Lebesgue measure, which is the unique measure on $\mathbb{R}^d$ with this property, cf. Prop. 10.4.II in [DVJ88]. Thus,

$$M_{\Xi,S}(\cdot) = S_V \nu(\cdot).$$

The intensity S_V of S_Ξ is called the *specific surface* of the stationary RACS Ξ. It gives the mean surface area per unit volume. In the following, we will always assume $S_V < \infty$.

Remark:
For general random closed sets with values not necessarily in the extended convex ring, the surface measure S_Ξ can be defined via

$$S_\Xi(A) := \mathscr{H}^{d-1}(\partial\Xi \cap A)$$

for any bounded Borel set $A \in \mathscr{B}(\mathbb{R}^d)$, provided the surface $\partial\Xi$ of Ξ is sufficiently smooth.

The *second moment measure* of S_Ξ is the measure on $\mathbb{R}^{2d}$ defined by

$$M^{(2)}_{\Xi,S}(A \times B) := \mathbb{E} S_\Xi(A) S_\Xi(B)$$

for bounded Borel sets $A, B \in \mathscr{B}(\mathbb{R}^d)$. It can be shown that the Campbell theorem holds for the second moment measure, cf. [DVJ88], p.188.

$M^{(2)}_{\Xi,S}$ does not necessarily take on finite values, even on bounded sets, but we will assume it is locally finite in the following. This is certainly the case if Ξ satisfies the integrability condition

$$\mathbb{E} 4^{N(\Xi \cap K)} < \infty \quad \text{for any } K \in \mathcal{K}, \tag{2.1}$$

where $N(K)$ denotes, for any $K \in \mathcal{R}$, the minimal number $n \in \mathbb{N}$ such that K has a representation of the form

$$K = K_1 \cup \ldots \cup K_n, \quad K_1, \ldots, K_n \in \mathcal{K}.$$

This can be seen as follows.

Let $A, B \in \mathscr{B}(\mathbb{R}^d)$ be bounded and $K, K_0 \in \mathcal{K}$ such that $A, B \subset \text{int}K$ and $K \subset K_0$. Then the intersection $\Xi \cap K \in \mathcal{R}$. Let

$$\Xi \cap K = \bigcup_{i=1}^{N(\Xi \cap K)} K_i, \quad K_i \in \mathcal{K},$$

be a minimal representation of $\Xi \cap K$ as a union of convex bodies. We can assume without loss of generality that

$$\big|\Phi_{d-1}(\Xi, A)\big| \geq \big|\Phi_{d-1}(\Xi, B)\big|.$$

Then, using the additivity of Φ_{d-1},

$$\begin{aligned}
\mathbb{E}\big|\Phi_{d-1}(\Xi, A)\Phi_{d-1}(\Xi, B)\big| &= \mathbb{E}\big|\Phi_{d-1}(\Xi \cap K, A)\Phi_{d-1}(\Xi \cap K, B)\big| \\
&\leq \mathbb{E}\big|\Phi_{d-1}(\Xi \cap K, A)\big|^2 \\
&= \mathbb{E}\Big|\sum_{v \in S(N(\Xi \cap K))} (-1)^{|v|-1}\Phi_{d-1}(K_v, A)\Big|^2 \\
&\leq \mathbb{E}\left(\sup_{\substack{K' \in \mathcal{K} \\ K' \subset K_0}} \big|\Phi_{d-1}(K', A)\big| 2^{N(\Xi \cap K_0)}\right)^2 \\
&= \sup_{\substack{K' \in \mathcal{K} \\ K' \subset K_0}} \big|\Phi_{d-1}(K', A)\big|^2 \mathbb{E}4^{N(\Xi \cap K_0)}.
\end{aligned}$$

The last expectation is finite if Ξ satisfies (2.1), which yields the assumption. Clearly, this condition also ensures the finiteness of $M_{\Xi,S}$.

Similar to $M_{\Xi,S}$, $M^{(2)}_{\Xi,S}$ also has an important invariance property due to the stationarity of S_Ξ. For $x \in \mathbb{R}^d$ and bounded Borel sets $A, B \in \mathscr{B}(\mathbb{R}^d)$ we have

$$\begin{aligned}
M^{(2)}_{\Xi,S}((A + x) \times (B + x)) &= \mathbb{E}S_\Xi(A + x)S_\Xi(B + x) \\
&= \mathbb{E}T_x S_\Xi(A)T_x S_\Xi(B) \\
&= \mathbb{E}S_\Xi(A)S_\Xi(B) \\
&= M^{(2)}_{\Xi,S}(A \times B)
\end{aligned}$$

since $\mathbb{P}_{T_x S_\Xi} = \mathbb{P}_{S_\Xi}$. As the rectangular Borel sets $A \times B$ generate the product σ-algebra in $\mathbb{R}^{2d}$, $M^{(2)}_{\Xi,S}$ is invariant under the diagonal shifts D_x defined for any $x \in \mathbb{R}^d$ and $(y_1, y_2) \in \mathbb{R}^{2d}$ by

$$D_x(y_1, y_2) = (y_1 + x, y_2 + x).$$

We are now in a position to state an important property of the surface measure S_Ξ.

Definition 2.1.1 A locally finite random measure is *second order stationary* if

- its first moment measure exists as a locally finite measure and is translation invariant, and

- its second moment measure exists as a locally finite measure and is invariant under diagonal shifts.

Hence, we can formulate the following

Lemma 2.1.1 Let Ξ be a stationary random $\mathcal{S}$-set satisfying (2.1). Then the surface measure S_Ξ of Ξ is second order stationary.

Closely related to the second moment measure is the covariance measure of S_Ξ. It is defined as

$$\begin{aligned}\mathrm{Cov}_{\Xi,S}(A \times B) &:= \mathrm{cov}(S_\Xi(A), S_\Xi(B)) \\ &= M^{(2)}_{\Xi,S}(A \times B) - M_{\Xi,S}(A)M_{\Xi,S}(B).\end{aligned}$$

The covariance measure inherits diagonal shift invariance from $M^{(2)}_{\Xi,S}$.

Due to diagonal shift invariance, $M^{(2)}_{\Xi,S}$ and $\mathrm{Cov}_{\Xi,S}$ factorize into a Lebesgue part along the diagonal $y_1 = y_2$ and a reduced measure along the rest of the space $\mathbb{R}^{2d}$, cf. [DVJ88], Lemma 10.4.III. That is, for any bounded measurable function f of compact support, we have,

$$\int\limits_{\mathbb{R}^{2d}} f(x_1, x_2) M^{(2)}_{\Xi,S}(\mathrm{d}x_1, \mathrm{d}x_2) = \int\limits_{\mathbb{R}^d} \mathrm{d}x \int\limits_{\mathbb{R}^d} f(x, x+h) \hat{M}^{(2)}_{\Xi,S}(\mathrm{d}h) \tag{2.2}$$

and

$$\int\limits_{\mathbb{R}^{2d}} f(x_1, x_2) \mathrm{Cov}_{\Xi,S}(\mathrm{d}x_1, \mathrm{d}x_2) = \int\limits_{\mathbb{R}^d} \mathrm{d}x \int\limits_{\mathbb{R}^d} f(x, x+h) \hat{\mathrm{Cov}}_{\Xi,S}(\mathrm{d}h) \tag{2.3}$$

where $\hat{M}^{(2)}_{\Xi,S}$ and $\hat{\mathrm{Cov}}_{\Xi,S}$ are symmetric, locally finite measures. They are called the *reduced second moment measure* and the *reduced covariance measure*, respectively. We will be concerned mainly with the (reduced) second moment measure, but since in spectral analysis quantities in frequency space are, by convention, associated with covariances, we include the covariance measure here for clarity.

If the RACS Ξ is isotropic as well as stationary, then the second order structure of the measure S_Ξ can be summarized by the reduced second moment function $K(\cdot)$ defined by

$$K(r) := S_V^{-2} \hat{M}^{(2)}_{\Xi,S}(B_r(0)), \quad r \geq 0.$$

The quantity $S_V K(r)$ can be interpreted as the mean area of all surface pieces contained in a ball of radius r centred at a 'typical' surface point of Ξ, cf. [SKM95], p.302.

The second moment measure $M^{(2)}_{\Xi,S}$ can be absolutely continuous with respect to Lebesgue measure. Then its density

$$m^{(2)}(x,y), \quad x,y \in \mathbb{R}^d,$$

is called the *surface product density* of S_Ξ. The density of the reduced second moment measure $\hat{M}^{(2)}_{\Xi,S}$ in this case is related to $m^{(2)}$ via

$$m(h) = m^{(2)}(x, x+h), \quad x,h \in \mathbb{R}^d,$$

cf. [DVJ88], Prop.10.4.V. In materials science, the function m is known as the two-point surface-surface correlation function; we will refer to it as *surface correlation function* for short. Moreover, the density cov_S of the reduced covariance measure, called the *surface covariance function*, is given by

$$\mathrm{cov}_S(x) = m(x) - S_V^2, \quad x \in \mathbb{R}^d.$$

The subscript S denotes the relation to the surface measure. Quite similarly, the covariance function $\mathrm{cov}_{V,0}$ arises as the density of the reduced covariance measure $\hat{\mathrm{Cov}}_{\Xi,V}$ associated with the volume measure V_Ξ of a stationary RACS Ξ, cf. [Koc02].

If they exist, the densities m and cov_S contain all the second order information related to the surface of the stationary RACS Ξ. Therefore, they are indispensable quantities for the analysis of the second order surface structure of Ξ.

The absolute continuity of the second moment measure of S_Ξ has not received much consideration in the literature so far. Only recently, Ballani has proved the existence of the surface product density for certain classes of random $\mathcal{S}$-sets, cf. [Bal05]. Every random closed set whose realizations lie a.s. in the extended convex ring and that satisfies

$$\mathbb{E}N(\Xi \cap C) < \infty$$

for all $C \in \mathcal{C}'$ can be represented as the union set of a point process Ψ on the product space $\mathbb{R}^d \times \mathcal{K}$, cf. [SW00], Satz 4.4.2. If the intensity measure of Ψ is given by

$$\Lambda(\mathrm{d}(x,K)) = f(x,K)\mathrm{d}x\mathbb{Q}(\mathrm{d}K),$$

then the absolute continuity of $M^{(2)}_{\Xi,S}$ basically depends on the mark distribution $\mathbb{Q}$. Ballani has shown that the product density exists whenever the K_i are strictly convex and whenever the mark distribution is rotation invariant, *loc.cit.*

However, there are also examples where $M^{(2)}_{\Xi,S}$ is not absolutely continuous. This is the case if there exists a set of grains that has positive $\mathbb{Q}$-measure and whose elements all have a planar surface part with outer normal vector u. As an example for such a RACS, one can think of a union of cuboids that are all parallel, *loc.cit.*

To conclude this section, we will give two examples of stationary isotropic RACS for which the surface product densities exist and can be calculated explicitly.

Example 2.1.1
Let Ξ be a stationary isotropic Boolean model in $\mathbb{R}^d$, $d \geq 2$, whose primary grains are spherical shells of radius R and let λ denote the intensity of the underlying Poisson point process. Ξ is not an $\mathcal{S}$-set since its grains are not convex. Its surface is sufficiently smooth, however, to allow for the definition of the surface measure of Ξ via

$$S_\Xi(A) := \mathscr{H}^{d-1}(\partial\Xi \cap A)$$

for bounded Borel sets $A \in \mathscr{B}(\mathbb{R}^d)$.

The specific surface is given by

$$S_V = d\lambda\kappa_d R^{d-1}.$$

By isotropy it suffices to consider the reduced second moment function $K(r)$ of Ξ. As mentioned above, the quantity $S_V K(r)$ can be interpreted as the mean area of surface pieces within a ball of radius r centred at a 'typical' surface point.

By an analogon of the theorem of Slivnyak, cf. [SKM95], the part of the Boolean model not containing the shell holding the 'typical' point has the same distributional properties as the original model, so that the contribution of these shells is given by $\mathbb{E}S_\Xi(B_r(0)) = S_V\nu(B_r(0))$. The contribution of the shell on which the 'typical' point is located is the area of the spherical cap cut out of this shell by the ball of radius r centred at the 'typical' point.

Hence, $S_V K(r)$ can be evaluated to

$$S_V K(r) = S_V\kappa_d r^d + F(r), \quad r \geq 0,$$

where

$$F(r) = \begin{cases} R^{d-1}\omega_{d-1} \int_0^{\arccos(1-\frac{r^2}{2R^2})} \sin^{d-2}(\varphi)d\varphi, & r < 2R, \\ R^{d-1}\omega_d, & r \geq 2R, \end{cases}$$

is the area of the spherical cap. For a derivation of $F(\cdot)$, see Appendix A.

Clearly, $S_V K(r)$ is continuous in $r = 2R$ and hence differentiable. We can calculate the density m of the measure $\hat{M}^{(2)}_{\Xi,S}$ via

$$m(r) = \frac{S_V^2}{d\kappa_d r^{d-1}} \frac{\mathrm{d}}{dr} K(r).$$

Thus, we obtain

$$m(r) = S_V^2 + \frac{\lambda R^{d-1}}{r^{d-1}} \frac{\mathrm{d}}{dr} F(r), \quad r \geq 0.$$

Note that the density of the covariance measure,

$$\mathrm{cov}_S(r) = m(r) - S_V^2 = \frac{\lambda R^{d-1}}{r^{d-1}} \frac{\mathrm{d}}{dr} F(r),$$

is zero for $r \geq 2R$, since $\frac{\mathrm{d}}{dr}F(r) = 0$ for $r \geq 2R$.
For $d = 3$, we obtain the explicit results

$$m(r) = \begin{cases} S_V^2 + \frac{2\lambda\pi R^2}{r}, & r < 2R, \\ S_V^2, & r \geq 2R, \end{cases}$$

and

$$\mathrm{cov}_S(r) = \begin{cases} \frac{2\lambda\pi R^2}{r}, & r < 2R, \\ 0, & r \geq 2R. \end{cases}$$

Example 2.1.2
Another example of a stationary RACS for which the surface product density exists is the RACS Ξ obtained as the union set of a stationary isotropic Poisson hyperplane process Ψ in $\mathbb{R}^d$, where Ψ has locally finite intensity measure Λ. The intersection of Ξ with a convex body $K \in \mathcal{K}$ a.s. is a finite union of convex bodies, so that Ξ can be interpreted as an $\mathcal{S}$-set. Since such an intersection a.s. consists of subsets of hyperplanes, the surface measure can be interpreted to measure the part of Ξ contained in the bounded set $A \in \mathscr{B}(\mathbb{R}^d)$, that is,

$$S_\Xi(A) = \mathscr{H}^{d-1}(\partial\Xi \cap A) = \mathscr{H}^{d-1}(\Xi \cap A),$$

from which it follows that

$$S_\Xi(A) = \nu_{d-1}(\Xi \cap A) = \sum_{E \in \Psi} \nu_{d-1}(E \cap A),$$

since $\mathscr{H}^k = \nu_k$ *on any* k*-dimensional linear subspace of* $\mathbb{R}^d$*. This follows from Thm. 2.10.35 of [Fed69] and the last equality holds due to the additivity of* ν_{d-1}*. Since*

$$\mathbb{E} \sum_{E \in \Psi} \nu_{d-1}(E \cap A) = \lambda \nu(A)$$

for a stationary hyperplane process, the specific surface S_V *in this case is just the intensity* λ *of the process.*

The reduced second moment function $S_V K(r)$ *is given by*

$$S_V K(r) = S_V \kappa_d r^d + \kappa_{d-1} r^{d-1}, \quad r \geq 0,$$

which can be explained analogously as in the previous example. This formula is a generalization of the results for the two- and three-dimensional cases given in [SKM95], pp.285 and 302, respectively.

The density m *calculates to*

$$m(r) = S_V^2 + \frac{S_V \Gamma(\frac{d}{2})}{\pi \Gamma(\frac{d-1}{2}) r} \quad r \geq 0,$$

whence

$$\operatorname{cov}_S(r) = \frac{S_V \Gamma(\frac{d}{2})}{\pi \Gamma(\frac{d-1}{2}) r}, \quad r \geq 0.$$

Thus, the covariance density is > 0 *for all* $r \geq 0$ *in this case.*

2.1.2 The Bartlett Spectrum

As we have seen, the reduced covariance measure of a stationary random $\mathcal{S}$-set Ξ does not necessarily have a density. Even if it does, the density in general is not necessarily continuous nor an element of $L^p(\mathbb{R}^d)$, $1 \leq p \leq 2$. Hence, it may not be Fourier transformable. It is convenient, therefore, to consider the measure $\hat{\mathrm{Cov}}_{\Xi,S}$ itself. Since it is not totally finite in general, its Fourier-Stieltjes transform does not necessarily exist. Thus we must first define what is meant by a Fourier transform in the context of locally finite measures. Similar to the theory of generalized functions, the Parseval identity can be used for the definition, cf. [DVJ88], Def.11.1.I.

Definition 2.1.2 A locally finite signed measure μ on $\mathbb{R}^d$ is *transformable* if there exists a locally finite measure η on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} \tilde{\psi}(\xi)\mu(\mathsf{d}\xi) = \int_{\mathbb{R}^d} \psi(x)\eta(\mathsf{d}x) \tag{2.4}$$

holds for all $\psi \in \mathscr{S}(\mathbb{R}^d)$.
The measure η is called the *Fourier transform* of μ.

If μ is totally finite, then (2.4) is just the usual Parseval identity, cf. [SD80], part II, Satz 1.10. Hence, this definition extends the classical definition of the Fourier transformation on $L^1(\mathbb{R}^d)$.

The general proof of the transformability of the reduced covariance measure of a second order stationary random measure has been given in Prop. 11.2.I of [DVJ88]. We will follow their line of argumentation to establish the existence of a spectral measure associated with $\hat{\text{Cov}}_{\Xi,S}$.

Let φ be a bounded measurable function of compact support. Then

$$\begin{aligned}
0 &\leq \operatorname{var} \int_{\mathbb{R}^d} \varphi(x) S_\Xi(\mathrm{d}x) \\
&= \mathbb{E} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x_1)\overline{\varphi(x_2)} S_\Xi(\mathrm{d}x_1) S_\Xi(\mathrm{d}x_2) - S_V^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x_1)\overline{\varphi(x_2)}\mathrm{d}x_1\mathrm{d}x_2 \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x_1)\overline{\varphi(x_2)} Cov_\Xi(\mathrm{d}x_1, \mathrm{d}x_2) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)\overline{\varphi(x+h)}\mathrm{d}x \hat{\text{Cov}}_{\Xi,S}(\mathrm{d}h) \\
&\overset{symm.}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)\overline{\varphi(x-h)}\mathrm{d}x \hat{\text{Cov}}_{\Xi,S}(\mathrm{d}h) \\
&= \int_{\mathbb{R}^d} \varphi * \overline{\varphi}^*(h) \hat{\text{Cov}}_{\Xi,S}(\mathrm{d}h).
\end{aligned}$$

As an immediate consequence, we obtain

$$\int_{\mathbb{R}^d} \psi * \psi^*(x) \hat{\text{Cov}}_{\Xi,S}(\mathrm{d}x) \geq 0$$

for all bounded measurable ψ of compact support. A locally finite measure satisfying this inequality for all such ψ is called *positive-definite*. If we replace the variance above by the expectation, we can easily see that $\hat{M}^{(2)}_{\Xi,S}$ is also positive-definite. This property is essential in proving the transformability of $\hat{M}^{(2)}_{\Xi,S}$ and $\hat{\mathrm{Cov}}_{\Xi,S}$.

It is not difficult to see that the measure $\hat{M}^{(2)}_{\Xi,S}$ is nonnegative. This follows from the nonnegativity of S_Ξ and from the fact that for all bounded Borel sets $A \in \mathscr{B}(\mathbb{R}^d)$, $\hat{M}^{(2)}_{\Xi,S}(A)$ can be expressed as

$$\hat{M}^{(2)}_{\Xi,S}(A) = \mathbb{E} \int_{C^d} S_\Xi(A+x) S_\Xi(\mathrm{d}x),$$

where C^d denotes the d-dimensional unit cube. This equality can easily be deduced from (2.2) by putting $f(x_1, x_2) = \mathbb{1}_{C^d}(x_1)\mathbb{1}_A(x_2 - x_1)$, cf. Prop.10.4.V in [DVJ88]. Since $\hat{\mathrm{Cov}}_{\Xi,S}$ is the difference of the nonnegative measures $\hat{M}^{(2)}_{\Xi,S}$ and $S_V^2\nu$, it is a signed measure rather than a measure. Nonnegativity simplifies the following argumentation immensely, so we will restrict our attention to $\hat{M}^{(2)}_{\Xi,S}$ for the moment.

Let $A \in \mathcal{K}$ with $\nu(A) > 0$ and $o \in \mathrm{int}A$. The function

$$\gamma_A(x) = \mathbb{1}_A * \mathbb{1}^*_A(x)$$

is nonnegative, continuous and of compact support, and it is symmetric, because

$$\begin{aligned}
\gamma_A(x) &= \int_{\mathbb{R}^d} \mathbb{1}_A(y)\mathbb{1}_A(y-x)\mathrm{d}y \\
&\overset{u=y-x}{=} \int_{\mathbb{R}^d} \mathbb{1}_A(u+x)\mathbb{1}_A(u)\mathrm{d}u \\
&= \gamma_A(-x)
\end{aligned}$$

for all $x \in \mathbb{R}^d$. If $A = C^d$ is the d-dimensional unit cube,

$$\gamma_{C^d}(x) = \prod_{i=1}^{d} (1 - |x_i|)_+, \quad x \in \mathbb{R}^d,$$

which is the multivariate extension of the triangular probability density used as a smoothing function in probability theory.

Hence, we can use γ_A to obtain a smoothed version of the measure $\hat{M}^{(2)}_{\Xi,S}$. The convolution

$$\big(\gamma_A * \hat{M}^{(2)}_{\Xi,S}\big)(x) = \int \gamma_A(x-y)\hat{M}^{(2)}_{\Xi,S}(\mathrm{d}y)$$

is well-defined, since $\hat{M}^{(2)}_{\Xi,S}$ is locally finite, and the function

$$c(x) := \big(\gamma_A * \hat{M}^{(2)}_{\Xi,S}\big)(x)$$

is real-valued and continuous.

Let ψ be a bounded measurable function of compact support. Then, since with $A \in \mathcal{K}$ compact we also have $-A \in \mathcal{K}$ compact, $\psi * \mathbb{1}_{(-A)}(\cdot) = \psi * \mathbb{1}'_A(\cdot)$ is measurable, bounded and of compact support. We have

$$\begin{aligned}
\int \psi * \psi^*(x)c(x)\mathrm{d}x &= \iint \psi * \psi^*(x)\big(\gamma_A * \hat{M}^{(2)}_{\Xi,S}\big)(x)\mathrm{d}x \\
&= \iint \psi * \psi^*(x)\gamma_A(x-y)\hat{M}^{(2)}_{\Xi,S}(\mathrm{d}y)\mathrm{d}x \\
&= \int (\psi * \psi^*) * (\mathbb{1}_A * \mathbb{1}^*_A)'(y)\hat{M}^{(2)}_{\Xi,S}(\mathrm{d}y) \\
&= \int (\psi * \mathbb{1}'_A) * (\psi * \mathbb{1}'_A)^*(y)\hat{M}^{(2)}_{\Xi,S}(\mathrm{d}y),
\end{aligned}$$

which follows by standard calculations. But $\hat{M}^{(2)}_{\Xi,S}$ is positive-definite so that the last term on the right hand side is ≥ 0. Hence,

$$\int \psi * \psi^*(x)c(x)\mathrm{d}x \geq 0,$$

which is equivalent to the positive-definiteness of $c(\cdot)$, since $c(\cdot)$ is continuous. By Thm.1.4.1 in [Sas94], $c(\cdot)$ is bounded, say, by $K < \infty$.

Consider the function γ_A. Since A contains the origin in its interior, $\gamma_A(x)$ is bounded away from zero for $\|x\|$ sufficiently small. That is, for $\|x\| \leq h$, $\gamma_A(x) \geq k > 0$ for a suitable constant k. Hence, for a ball $B_h(x)$ with centre at some $x \in \mathbb{R}^d$, we obtain

$$\begin{aligned}
\hat{M}^{(2)}_{\Xi,S}(B_h(x)) &= \int_{\mathbb{R}^d} \mathbb{1}_{B_h(x)}(y)\hat{M}^{(2)}_{\Xi,S}(\mathrm{d}y) \\
&= k^{-1}\int_{\mathbb{R}^d} k\mathbb{1}_{B_h(0)}(y-x)\hat{M}^{(2)}_{\Xi,S}(\mathrm{d}y) \\
&\leq k^{-1}\int_{\mathbb{R}^d} \gamma_A(x-y)\hat{M}^{(2)}_{\Xi,S}(\mathrm{d}y) \\
&= k^{-1}c(x) \leq k^{-1}K < \infty,
\end{aligned}$$

where we have used the symmetry of γ_A and the positivity of $\hat{M}^{(2)}_{\Xi,S}$. Hence, for the given h, there exists a constant K_h such that $\hat{M}^{(2)}_{\Xi,S}(B_h(x)) \leq K_h$ for all $x \in \mathbb{R}^d$. Now consider the ball $B_{h'}(y)$, $y \in \mathbb{R}^d$, $h' \neq h$. $B_{h'}(y)$ can be covered by a finite number of translated versions of $B_h(x)$. From the positivity of $\hat{M}^{(2)}_{\Xi,S}$ it follows that $\hat{M}^{(2)}_{\Xi,S}(B_{h'}(y)) \leq K_{h'}$ for a suitable constant $K_{h'}$.

Definition 2.1.3 A locally finite signed measure μ on $\mathbb{R}^d$ is called *translation bounded* if for all $h > 0$ there exists a finite constant K_h such that

$$\left|\mu(B_h(x))\right| \leq K_h$$

for all $x \in \mathbb{R}^d$.

We have now established the following result:

Lemma 2.1.2 The reduced second moment measure associated with the surface measure of a stationary random $\mathcal{S}$-set satisfying (2.1) is translation bounded.

The same smoothing technique as above can be used to establish the transformability of the measures $\hat{M}^{(2)}_{\Xi,S}$ and $\hat{\mathrm{Cov}}_{\Xi,S}$. Since $\hat{M}^{(2)}_{\Xi,S}$ is translation bounded, the convolution $\big(g * \hat{M}^{(2)}_{\Xi,S}\big)$ is well-defined for any function $g \in L^1(\mathbb{R}^d)$, cf. Corollary 1.1 in [AGdL74]. Clearly, if the function g is continuous, then so is the convolution $\big(g * \hat{M}^{(2)}_{\Xi,S}\big)$.

A suitable smoothing function is given by any element of the family

$$f_\lambda(x) = \left(\frac{\lambda}{2}\right)^d e^{-\lambda \sum\limits_{j=1}^{d} |x_j|}, \quad \lambda > 0,$$

which is the multivariate extension of the double exponential density of probability theory. The function f_λ, $\lambda > 0$, is bounded and continuous, hence integrable,

$$\int\limits_{\mathbb{R}^d} f_\lambda(x)\mathrm{d}x = 1,$$

and its Fourier transform

$$\tilde{f}_\lambda(\xi) = \prod_{j=1}^{d} \frac{\lambda^2}{\lambda^2 + \xi_j^2}, \quad \xi \in \mathbb{R}^d,$$

is clearly symmetric, continuous and > 0 for all $\xi \in \mathbb{R}^d$, which will prove valuable in the following. For $\lambda < \infty$, we have

$$\int_{\mathbb{R}^d} \left|f_\lambda(x)\right|^2 \mathrm{d}x < \infty,$$

so that f_λ is positive-definite by Thm.1.9.12 in [Sas94].

Fix $\lambda < \infty$. The convolution $\big(f_\lambda * \hat{M}^{(2)}_{\Xi,S}\big)$ is well-defined and defines a continuous positive-definite function on $\mathbb{R}^d$, which follows from the positive-definiteness of f_λ and $\hat{M}^{(2)}_{\Xi,S}$.

Hence, we are now in a position to apply Bochner's theorem. This yields the existence of a finite measure G_λ such that

$$\big(f_\lambda * \hat{M}^{(2)}_{\Xi,S}\big)(x) = \frac{1}{\sqrt{2\pi}^d} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} G_\lambda(\mathrm{d}\xi).$$

Thus, for any integrable function k, we obtain, by an application of Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^d} k(x)\big(f_\lambda * \hat{M}^{(2)}_{\Xi,S}\big)(x)\mathrm{d}x &= \int_{\mathbb{R}^d} k(x)\frac{1}{\sqrt{2\pi}^d}\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} G_\lambda(\mathrm{d}\xi)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi}^d}\int_{\mathbb{R}^d} k(x)e^{i\langle x,\xi\rangle}\mathrm{d}x G_\lambda(\mathrm{d}\xi) \\
&= \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi}^d}\int_{\mathbb{R}^d} k(x)e^{-i\langle x,-\xi\rangle}\mathrm{d}x G_\lambda(\mathrm{d}\xi) \\
&= \int_{\mathbb{R}^d} \tilde{k}(-\xi) G_\lambda(\mathrm{d}\xi).
\end{aligned}$$

Now we define a measure G on $\mathbb{R}^d$ via

$$G(\mathrm{d}\xi) = \tilde{f}_\lambda(\xi)^{-1}\sqrt{2\pi}^{-d} G_\lambda(\mathrm{d}\xi).$$

Then G is locally finite, since G_λ is finite and $\tilde{f}_\lambda^{-1}$ is continuous. We will show that G thus defined satisfies the relation

$$\int_{\mathbb{R}^d} \tilde{\psi}(x)\hat{M}^{(2)}_{\Xi,S}(\mathrm{d}x) = \int_{\mathbb{R}^d} \psi(\xi)G(\mathrm{d}\xi)$$

for all $\psi \in \mathscr{S}$, which yields the transformability of $\hat{M}^{(2)}_{\Xi,S}$.

Let $\psi \in \mathscr{S}$, so that ψ has a Fourier transform $\tilde{\psi} \in \mathscr{S}$. The function h defined by

$$h(\xi) := \psi(-\xi)\tilde{f}_\lambda(\xi)^{-1}\sqrt{2\pi}^{-d}$$

is an element of $\mathscr{S}$. Hence, h is the Fourier transform of a function k, say, that is

$$\tilde{k}(\xi) = \psi(-\xi)\tilde{f}_\lambda(\xi)^{-1}\sqrt{2\pi}^{-d}.$$

For a function $\psi \in \mathscr{S}$, we have, for $\xi \in \mathbb{R}^d$,

$$\begin{aligned}\tilde{\tilde{\psi}}(\xi) &= \frac{1}{\sqrt{2\pi}^d}\int_{\mathbb{R}^d} \tilde{\psi}(x)e^{-i\langle x,\xi\rangle}\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}^d}\int_{\mathbb{R}^d} \tilde{\psi}(x)e^{i\langle x,-\xi\rangle}\mathrm{d}x \\ &= \psi(-\xi).\end{aligned}$$

Thus, since $\mathscr{F}(k * f_\lambda) = \sqrt{2\pi}^d\mathscr{F}(k)\cdot\mathscr{F}(f_\lambda)$, cf. [Kön00], ch. 10.2.II, it follows that k satisfies

$$\tilde{\psi}(\xi) = k * f_\lambda(\xi).$$

Furthermore, by multiple application of Fubini's theorem, we obtain

$$\begin{aligned}\int \tilde{\psi}(x)\hat{M}^{(2)}_{\Xi,S}(\mathrm{d}x) &= \int \big(k * f_\lambda\big)(x)\hat{M}^{(2)}_{\Xi,S}(\mathrm{d}x) \\ &= \iint k(y)f_\lambda(x-y)\mathrm{d}y\hat{M}^{(2)}_{\Xi,S}(\mathrm{d}x) \\ &\overset{f_\lambda symm.}{=} \iint k(y)f_\lambda(y-x)\hat{M}^{(2)}_{\Xi,S}(\mathrm{d}x)\mathrm{d}y \\ &= \int k(y)\big(f_\lambda * \hat{M}^{(2)}_{\Xi,S}\big)(y)\mathrm{d}y \\ &= \int \tilde{k}(-\xi)G_\lambda(\mathrm{d}\xi) \\ &\overset{\tilde{f}_\lambda symm.}{=} \int \psi(\xi)\tilde{f}_\lambda(\xi)^{-1}\sqrt{2\pi}^{-d}G_\lambda(\mathrm{d}\xi) \\ &= \int \psi(\xi)G(\mathrm{d}\xi).\end{aligned}$$

Since ψ was arbitrary, the equation holds for all $\psi \in \mathscr{S}$. Integrals with respect to the functions $\psi \in \mathscr{S}$ uniquely determine a locally finite measure, so that G is uniquely determined by $\hat{M}^{(2)}_{\Xi,S}$ and vice versa. Hence, $\hat{M}^{(2)}_{\Xi,S}$ is transformable.

It can now be deduced that the reduced covariance measure is also transformable. As Lebesgue measure is transformable, which can be shown by the same argumentation as for $\hat{M}^{(2)}_{\Xi,S}$, a locally finite measure Γ satisfying

$$\int_{\mathbb{R}^d} \tilde{\psi}(x)\hat{\mathrm{Cov}}_{\Xi,S}(\mathrm{d}x) = \int_{\mathbb{R}^d} \psi(\xi)\Gamma(\mathrm{d}\xi)$$

for all $\psi \in \mathscr{S}$ can be defined via the difference of the transforms of $\hat{M}^{(2)}_{\Xi,S}$ and the measure $S_V^2\nu$.

Theorem 2.1.1 Let Ξ be a stationary random $\mathcal{S}$-set satisfying (2.1) and let S_Ξ denote the corresponding surface measure. Then there exists a locally finite spectral measure Γ on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} \tilde{\psi}(x)\hat{\mathrm{Cov}}_{\Xi,S}(\mathrm{d}x) = \int_{\mathbb{R}^d} \psi(\xi)\Gamma(\mathrm{d}\xi)$$

for all $\psi \in \mathscr{S}$. The measure Γ is called the *Bartlett spectrum* of S_Ξ.

If the reduced covariance measure is absolutely continuous with respect to Lebesgue measure, that is, if its density cov_S on $\mathbb{R}^d$ exists, we have, for any bounded Borel set A,

$$\hat{\mathrm{Cov}}_{\Xi,S}(A) = \int_A \mathrm{cov}_S(x)\mathrm{d}x.$$

If, in addition, cov_S is absolutely integrable, then its Fourier transform $\tilde{\mathrm{cov}}_S$ exists. Then, for any $\psi \in \mathscr{S}$, the Parseval relation, cf. Satz 1.2 in part II of [SD80], yields

$$\begin{aligned}
\int_{\mathbb{R}^d} \tilde{\psi}(x)\hat{\mathrm{Cov}}_{\Xi,S}(\mathrm{d}x) &= \int_{\mathbb{R}^d} \tilde{\psi}(x)\mathrm{cov}_S(x)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \psi(\xi)\tilde{\mathrm{cov}}_S(\xi)\mathrm{d}\xi \\
&= \int_{\mathbb{R}^d} \psi(\xi)\Gamma(\mathrm{d}\xi).
\end{aligned}$$

Due to the determining properties of the elements of $\mathscr{S}$, it follows that the measures $\tilde{\mathrm{cov}}_S(\xi)\mathrm{d}\xi$ and $\Gamma(\mathrm{d}\xi)$ coincide.

Thus, if the density cov_S of the reduced covariance measure of a stationary random $\mathcal{S}$-set Ξ exists and is integrable, then the Bartlett spectrum Γ of Ξ is absolutely continuous with density $\tilde{\mathrm{cov}}_S$.

Example 2.1.3
Consider the Boolean model as in Example 2.1.1. The density of the reduced covariance measure is given by

$$\mathrm{cov}_S(r) = \begin{cases} \frac{\lambda R^{d-1}}{r^{d-1}} \sin^{d-2}(\arccos(1 - \frac{r^2}{2R^2})) \frac{1}{\sqrt{R^2 - \frac{r^2}{4}}}, & r < 2R, \\ 0, & r \geq 2R, \end{cases}$$

where $r = \|x\|$ is the distance to the origin of a point $x \in \mathbb{R}^d$, cf. Appendix A.

As a function of $x \in \mathbb{R}^d$, $\mathrm{cov}_S(x)$ is integrable over $\mathbb{R}^d$. Hence, the density of the Bartlett spectrum is given by $\tilde{\mathrm{cov}}_S$.

Since cov_S is a radial function, its Fourier transform also is a function of the distance $\|\xi\|$ to the origin of $\xi \in \mathbb{R}^d$. By Satz 4.1 in part II of [SD80], for $d \geq 2$, it can be expressed as an integral involving the Bessel function $\mathfrak{J}_{\frac{d-1}{2}}$ of order $\frac{d-1}{2}$, that is,

$$\tilde{\mathrm{cov}}_S(\xi) = \frac{1}{\|\xi\|^{\frac{d-1}{2}}} \int_0^\infty \mathrm{cov}_S(r) r^{\frac{d}{2}} \mathfrak{J}_{\frac{d-2}{2}}(r\|\xi\|) dr$$

for every $\xi \in \mathbb{R}^d$.

Example 2.1.4
In contrast to the previous example, we find a different situation in Example 2.1.2. The covariance density

$$\mathrm{cov}_S(x) = \frac{S_V \Gamma(\frac{d}{2})}{\pi \Gamma(\frac{d-1}{2}) \|x\|}, \quad x \in \mathbb{R}^d,$$

of the union set Ξ of a stationary, isotropic Poisson hyperplane process is integrable over the ball $B_R(0)$ for any radius $R < \infty$, which follows from Bsp. 8.2.II(i) in [Kön00]. But since

$$\int_{B_R(0)} \mathrm{cov}_S(x) dx = S_V \frac{\Gamma(\frac{d}{2})}{\pi \Gamma(\frac{d-1}{2})} \kappa_d R^d \to \infty \quad (R \to \infty),$$

the covariance density is not integrable over $\mathbb{R}^d$. Hence, the Bartlett spectrum does not have a density in this case.

2.2 Wiener-Khintchine Type Theorem

Consider a function $f \in L^2(\mathbb{R})$ interpreted as a signal depending on time. The second order structure of the signal is described by its autocorrelation function. Hence, if the Fourier transform of the autocorrelation function exists, it will provide the same second order information as a function in frequency space. In practical applications, such as physics and materials science, often spectral quantities are more easily interpretable than quantities given in the original space, therefore it is desirable to find spectral representations of second order quantities. The Wiener-Khintchine theorem provides the necessary means.

For a function $f \in L^2(\mathbb{R})$, the L^2 norm

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}} |f(t)|^2 \mathrm{d}t$$

is called the energy norm. Thus

$$E_f := \|f\|_{L^2}^2$$

describes the total energy of the function. By the Plancherel theorem, in $L^2(\mathbb{R})$ we have

$$\int_{\mathbb{R}} |f(t)|^2 \mathrm{d}t = \int_{\mathbb{R}} |\tilde{f}(\xi)|^2 \mathrm{d}\xi$$

so that $|\tilde{f}(\xi)|^2$ describes the distribution of energy as a function of frequency. It is called *energy density spectrum*. The Wiener-Khintchine theorem, cf. [MP96], p.299, states that the energy density spectrum is equal to the Fourier transform of the autocorrelation function associated with the signal.

In the higher-dimensional context, the energy density spectrum of $f \in L^2(\mathbb{R}^d)$ can be interpreted as a function of spatial frequency.

For a stationary RACS Ξ and a compact set $W \subset \mathbb{R}^d$ of positive volume, the energy density spectrum of Ξ with respect to W is defined by

$$\mathrm{eds}_W(x) = \mathbb{E}|\tilde{f}(x)|^2$$

where the function

$$f(x) = \mathbb{1}_W(x)(\mathbb{1}_\Xi(x) - p)$$

is defined via the stochastic field $\mathbb{1}_\Xi$ corresponding to Ξ. In [Koc02], an analogon of the theorem of Wiener-Khintchine has been established relating eds$_W$ to the covariance function cov$_{V,0}$ of Ξ, which arises as the density of the reduced covariance measure associated with the volume measure V_Ξ of Ξ. Furthermore, a relation between the paircorrelation function of a stationary simple point process Ψ and the energy density spectrum of the random function X_W has been established, where X_W is the restriction to W of the stochastic process X defined as the convolution of the mean corrected process $(\Psi - \Lambda)$ with a suitable kernel function. Here, Λ denotes the intensity measure of Ψ.

Since the characteristic function $\mathbb{1}_{\partial\Xi}$ of the boundary $\partial\Xi$ of a stationary random $\mathcal{S}$-set Ξ is a.s. equal to zero, the definition of the energy density spectrum of $\partial\Xi$ is of no use. Instead, it is more convenient to consider a smoothed version of the surface measure S_Ξ. By convolving the surface measure with a suitable kernel function, a stochastic process can be defined. Then the covariance function of the process is related to the second order densities of S_Ξ, so that a relation of the Wiener-Khintchine type will carry implications on these quantities.

In the following, let Ξ be a stationary random $\mathcal{S}$-set with surface measure S_Ξ and locally finite first moment measure $M_{\Xi,S}$. Let κ be a nonnegative function from $\mathbb{R}^d$ to $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}^d} \kappa(x)\mathrm{d}x = 1.$$

Note that the function $\kappa * \kappa^*$ then is symmetric, which can be shown similarly as for γ_A in the previous section, and satisfies

$$\int_{\mathbb{R}^d} \kappa * \kappa^*(x)\mathrm{d}x = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \kappa(y)\kappa(y-x)\mathrm{d}y\mathrm{d}x \overset{Fubini}{=} \int_{\mathbb{R}^d} \kappa(x)\mathrm{d}x \int_{\mathbb{R}^d} \kappa(y)\mathrm{d}y = 1$$

due to the translation invariance of Lebesgue measure. Moreover, if κ is of compact support, then so is $\kappa * \kappa^*$.

We define the stochastic process X^0 on $\mathbb{R}^d$ via

$$\begin{aligned} X^0 : \Omega \times \mathbb{R}^d &\rightarrow \mathbb{R} \\ (\omega, x) &\mapsto \big((S_\Xi - M_{\Xi,S}) * \kappa\big)(x). \end{aligned}$$

The mapping is well-defined and measurable, which follows from the Campbell theorem for random measures.

We have

$$\begin{aligned}
\mathbb{E}X^0(x) &= \mathbb{E}\big((S_\Xi - M_{\Xi,S}) * \kappa\big)(x) \\
&= \mathbb{E}\Big(\big(S_\Xi * \kappa\big)(x) - \big(M_{\Xi,S} * \kappa\big)(x)\Big) \\
&= \mathbb{E}\int_{\mathbb{R}^d} \kappa(x-y) S_\Xi(\mathrm{d}y) - \int_{\mathbb{R}^d} \kappa(x-u) M_{\Xi,S}(\mathrm{d}u) \\
&= \int_{\mathbb{R}^d} \kappa(x-y) M_{\Xi,S}(\mathrm{d}y) - \int_{\mathbb{R}^d} \kappa(x-u) M_{\Xi,S}(\mathrm{d}u) \\
&= 0
\end{aligned}$$

for $x \in \mathbb{R}^d$ by an application of the Campbell theorem and, in particular,

$$\big(M_{\Xi,S} * \kappa\big)(x) = \int_{\mathbb{R}^d} \kappa(x-y) M_{\Xi,S}(\mathrm{d}y) = S_V \int_{\mathbb{R}^d} \kappa(x-y)\mathrm{d}y = S_V, \quad x \in \mathbb{R}^d.$$

The covariance function of the process X^0 is defined by

$$\begin{aligned}
\mathrm{cov}_{X^0} : \mathbb{R}^d \times \mathbb{R}^d &\to \mathbb{R} \\
(x,y) &\mapsto \mathbb{E}\big(X^0(x) - \mathbb{E}X^0(x)\big)\big(X^0(y) - \mathbb{E}X^0(y)\big).
\end{aligned}$$

For $x, y \in \mathbb{R}^d$, we then obtain

$$\begin{aligned}
\mathrm{cov}_{X^0}(x,y) &= \mathbb{E}X^0(x)X^0(y) \\
&= \mathbb{E}\left(\int_{\mathbb{R}^d} \kappa(x-s) S_\Xi(\mathrm{d}s) - S_V\right)\left(\int_{\mathbb{R}^d} \kappa(y-t) S_\Xi(\mathrm{d}t) - S_V\right) \\
&= \mathbb{E}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \kappa(x-s)\kappa(y-t) S_\Xi(\mathrm{d}s) S_\Xi(\mathrm{d}t) \\
&\quad - S_V \mathbb{E}\int_{\mathbb{R}^d} \kappa(x-s) S_\Xi(\mathrm{d}s) - S_V \mathbb{E}\int_{\mathbb{R}^d} \kappa(y-t) S_\Xi(\mathrm{d}t) + S_V^2 \\
&= \int_{\mathbb{R}^{2d}} \kappa(x-s)\kappa(y-t) M^{(2)}_{\Xi,S}(\mathrm{d}s,\mathrm{d}t) - S_V^2 \\
&= \int_{\mathbb{R}^{2d}} \kappa(x-s)\kappa(y-t) \mathrm{Cov}_{\Xi,S}(\mathrm{d}s,\mathrm{d}t), \qquad (2.5)
\end{aligned}$$

where we have used the fact that the Campbell theorem holds also for the second moment measure.

The process X^0 is not necessarily integrable, hence its energy density spectrum does not necessarily exist. But we can obtain a local result in the following way: if we choose κ such that it decreases sufficiently fast for $\|x\| \to \infty$, we can make sure that X^0 is a.s. locally integrable. If we have κ such that for a.e. $x \in \mathbb{R}^d$

$$\big(S_\Xi * \kappa\big)(x) = \int_{\mathbb{R}^d} \kappa(x-y) S_\Xi(\mathrm{d}y) \leq K < \infty \quad \text{a.s.},$$

that is, such that $\big(S_\Xi * \kappa\big)$ is a.s. essentially bounded, then $\big(S_\Xi * \kappa\big)$ is a.s. integrable on every compact subset of $\mathbb{R}^d$ and since $\big(M_{\Xi,S} * \kappa\big)(x) < \infty$ for all $x \in \mathbb{R}^d$, it follows that the process X^0 is a.s. locally integrable.

If, in particular, κ is bounded and of compact support, then $\big(S_\Xi * \kappa\big)$ will be a.s. essentially bounded since S_Ξ is locally finite. Since in practical applications the data set under consideration will always be finite, the conditions on κ constitute no restriction. Thus, we can state the following

Lemma 2.2.1 Let Ξ be a stationary random $\mathcal{S}$-set with surface measure S_Ξ and locally finite first moment measure $M_{\Xi,S}$. Let

$$\kappa : \mathbb{R}^d \to \mathbb{R}$$

be a nonnegative bounded function of compact support satisfying

$$\int_{\mathbb{R}^d} \kappa(x)\mathrm{d}x = 1.$$

Then the restriction of the process

$$X^0(x) = \big((S_\Xi - M_{\Xi,S}) * \kappa\big)(x)$$

to a compact set $W \subset \mathbb{R}^d$ with $\nu(W) > 0$ yields a random function

$$X^0_W(x) := X^0(x)\mathbb{1}_W(x) = \big((S_\Xi - M_{\Xi,S}) * \kappa(x)\big)\mathbb{1}_W(x), \quad x \in \mathbb{R}^d,$$

which is a.s. integrable.

We will now prove the following analogon of the theorem of Wiener-Khintchine. For a compact set $W \subset \mathbb{R}^d$ with $\nu(W) > 0$,

$$\gamma_W(x) = \mathbb{1}_W * \mathbb{1}^*_W(x) = \nu(W \cap (W - x)).$$

The set W can be interpreted as a sampling window, so that γ_W is called the *window function* in this context.

Theorem 2.2.1 Let Ξ be a stationary random closed set with surface measure S_Ξ and locally finite first moment measure $M_{\Xi,S} = s\nu$ and reduced covariance measure $\hat{\mathrm{Cov}}_{\Xi,S}$. Assume that $\hat{\mathrm{Cov}}_{\Xi,S}$ has a density cov_S with respect to Lebesgue measure. Furthermore, let

$$\kappa : \mathbb{R}^d \to \mathbb{R}$$

be a bounded nonnegative function of compact support satisfying

$$\int_{\mathbb{R}^d} \kappa(x)\mathrm{d}x = 1.$$

If $X^0_W(x)$ denotes the restriction of the process $X^0(x) = \big((S_\Xi - M_{\Xi,S}) * \kappa\big)(x)$ to the compact set $W \subset \mathbb{R}^d$ with $\nu(W) > 0$, then

$$\sqrt{2\pi}^d \mathbb{E}|\tilde{X}^0_W(\xi)|^2 = \mathscr{F}(\gamma_W \cdot \mathrm{cov}_{X^0})(\xi), \quad \xi \in \mathbb{R}^d. \tag{2.6}$$

Proof:
Since S_Ξ is stationary, $\mathrm{Cov}_{\Xi,S}$ can be expressed via the reduced covariance measure $\hat{\mathrm{Cov}}_{\Xi,S}$. Then (2.5) becomes

$$\begin{aligned}\mathrm{cov}_{X^0}(x,y) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \kappa(x-r)\kappa(y-r-h)\mathrm{d}r\hat{\mathrm{Cov}}_{\Xi,S}(\mathrm{d}h)\\ &\overset{u=y-r}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \kappa(x-y+u)\kappa(u-h)\mathrm{d}u\hat{\mathrm{Cov}}_{\Xi,S}(\mathrm{d}h).\end{aligned}$$

Hence, in this case, cov_{X^0} is a function only of the difference of its arguments. Putting $v := x - y$ yields

$$\mathrm{cov}_{X^0}(v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \kappa(v+u)\kappa(u-h)\mathrm{d}u\hat{\mathrm{Cov}}_{\Xi,S}(\mathrm{d}h).$$

Moreover, since $\hat{\mathrm{Cov}}_{\Xi,S}$ is absolutely continuous by assumption, we can write $\hat{\mathrm{Cov}}_{\Xi,S}(\mathrm{d}h) = \mathrm{cov}_S(h)\mathrm{d}h$, so that, for $v \in \mathbb{R}^d$,

$$\mathrm{cov}_{X^0}(v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \kappa(v+u)\kappa(u-h)\mathrm{cov}_S(h)\mathrm{d}h\mathrm{d}u. \tag{2.7}$$

The reduced covariance measure is finite on compact subsets of $\mathbb{R}^d$, hence the function cov_S is locally integrable. Therefore, since the kernel function

κ has compact support, the integral in (2.7) is finite.

Now consider the process X_W^0. We have $X_W^0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ a.s. Hence, the Fourier transform $\tilde{X}_W^0$ exists and is an element of $L^2(\mathbb{R}^d)$ a.s., so that $|\tilde{X}_W^0|^2$ is a.s. integrable. By the Campbell theorem for random measures, it follows that its expectation exists and we obtain, for $\xi \in \mathbb{R}^d$,

$$\begin{aligned}
\mathbb{E}|\tilde{X}_W^0(\xi)|^2 &= \mathbb{E}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} X_W^0(x)e^{-i\langle\xi,x\rangle}\mathrm{d}x\int_{\mathbb{R}^d} X_W^0(y)e^{i\langle\xi,y\rangle}\mathrm{d}y \\
&= \mathbb{E}\frac{1}{(2\pi)^d}\int_W\int_W X^0(x)X^0(y)e^{-i\langle\xi,x-y\rangle}\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathrm{cov}_{X^0}(x,y)\mathbb{1}_W(x)\mathbb{1}_W(y)e^{-i\langle\xi,x-y\rangle}\mathrm{d}x\mathrm{d}y \\
&\overset{v=x-y}{=} \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathrm{cov}_{X^0}(v)\mathbb{1}_W(x)\mathbb{1}_W(x-v)e^{-i\langle\xi,v\rangle}\mathrm{d}x\mathrm{d}v \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{cov}_{X^0}(v)\gamma_W(v)e^{-i\langle\xi,v\rangle}\mathrm{d}v \\
&= \frac{1}{\sqrt{2\pi}^d}\mathscr{F}(\mathrm{cov}_{X^0}\cdot\gamma_W)(\xi)
\end{aligned}$$

by an application of Fubini's theorem, which yields the assumption. □

We can now examine the relationship between $\mathbb{E}|\tilde{X}_W^0(\xi)|^2$ and the second order quantities associated with the surface measure S_Ξ. Equation (2.7) can be transformed to

$$\begin{aligned}
\mathrm{cov}_{X^0}(v) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\kappa(v+u)\kappa(u-h)\mathrm{cov}_S(h)\mathrm{d}u\mathrm{d}h \\
&\overset{y=v+u}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\kappa(y)\kappa(y-(v+h))\mathrm{cov}_S(h)\mathrm{d}y\mathrm{d}h \\
&\overset{x=v+h}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\kappa(y)\kappa(y-x)\mathrm{cov}_S(x-v)\mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbb{R}^d}\kappa*\kappa^*(x)\mathrm{cov}_S(x-v)\mathrm{d}x
\end{aligned}$$

for $v \in \mathbb{R}^d$. By definition, the covariance function of the process X^0 is symmetric and the function $\kappa * \kappa^*$ has the same property so that

$$\begin{aligned}
\mathrm{cov}_{X^0}(v) &= \mathrm{cov}_{X^0}(-v) \\
&= \int_{\mathbb{R}^d} \kappa * \kappa^*(-h)\mathrm{cov}_S(v-h)\mathrm{d}h \\
&= \int_{\mathbb{R}^d} \kappa * \kappa^*(h)\mathrm{cov}_S(v-h)\mathrm{d}h \\
&= (\kappa * \kappa^*) * \mathrm{cov}_S(v) \qquad (2.8)
\end{aligned}$$

for $v \in \mathbb{R}^d$. Hence, we can write

$$\sqrt{2\pi}^d \mathbb{E}|\tilde{X}^0_W(\xi)|^2 = \mathscr{F}(\gamma_W \cdot ((\kappa * \kappa^*) * \mathrm{cov}_S))(\xi), \quad \xi \in \mathbb{R}^d.$$

As the function γ_W is of compact support, the function $\gamma_W \cdot ((\kappa * \kappa^*) * \mathrm{cov}_S)$ is an element of $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. We have $\mathscr{F}(L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)) \subset L^2(\mathbb{R}^d)$ and since the definitions of the Fourier transforms on $L^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ coincide on $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we may apply the Fourier cotransformation to the above equation, which gives

$$\sqrt{2\pi}^d \overline{\mathscr{F}}(\mathbb{E}|\tilde{X}^0_W|^2)(x) = (\gamma_W \cdot ((\kappa * \kappa^*) * \mathrm{cov}_S))(x), \quad x \in \mathbb{R}^d.$$

If we choose W such that $\gamma_W(x) > 0$, which is certainly true for all $x \in \mathrm{int}W$ whenever $0 \in \mathrm{int}W$, we may state the following result:

Corollary 2.2.1 Assume the conditions of Theorem 2.2.1 are fulfilled. In addition, let the compact set $W \subset \mathbb{R}^d$ be such that it contains the origin in its interior. With the notations as given in Theorem 2.2.1,

$$(\kappa * \kappa^*) * \mathrm{cov}_S(x) = \sqrt{2\pi}^d \frac{\overline{\mathscr{F}}(\mathbb{E}|\tilde{X}^0_W|^2)(x)}{\gamma_W(x)}, \quad x \in \mathrm{int}W. \qquad (2.9)$$

Note that the function cov_S is locally integrable if, and only if, the surface correlation function m is locally integrable. Now if we express cov_S as

$$\mathrm{cov}_S(x) = m(x) - S_V^2, \quad x \in \mathbb{R}^d,$$

equation (2.8) becomes

$$\mathrm{cov}_{X^0}(x) = (\kappa * \kappa^*) * m(x) - S_V^2, \quad x \in \mathbb{R}^d,$$

so that $\gamma_W \cdot \mathrm{cov}_{X^0}$ is the sum of two functions in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Using the additivity of the Fourier transformation on $L^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$, we obtain, for $\xi \in \mathbb{R}^d$,

$$\begin{aligned}\sqrt{2\pi}^d \mathbb{E}|\tilde{X}_W^0(\xi)|^2 &= \mathscr{F}(\gamma_W \cdot \mathrm{cov}_{X^0})(\xi) \\ &= \mathscr{F}(\gamma_W \cdot ((\kappa * \kappa^*) * m))(\xi) - \mathscr{F}(S_V^2 \cdot \gamma_W)(\xi),\end{aligned}$$

where every summand is an element of $L^2(\mathbb{R}^d)$. Application of the inverse Fourier transformation to each of the summands yields

$$\sqrt{2\pi}^d \overline{\mathscr{F}}(\mathbb{E}|\tilde{X}_W^0|^2)(x) = (\gamma_W \cdot ((\kappa * \kappa^*) * m))(x) - S_V^2 \gamma_W(x), \quad x \in \mathbb{R}^d,$$

so that (2.9) becomes

$$(\kappa * \kappa^*) * m(x) = \sqrt{2\pi}^d \frac{\overline{\mathscr{F}}(\mathbb{E}|\tilde{X}_W^0|^2)(x)}{\gamma_W(x)} - S_V^2, \quad x \in \mathrm{int}W.$$

2.2.1 Estimating the Surface Correlation Function

In order to construct an estimator for the surface correlation function m, it is convenient to consider the process

$$X(x) = S_\Xi * \kappa(x)$$

instead of the zero-mean process $X^0(x) = (S_\Xi - M_{\Xi,S}) * \kappa(x)$. Its covariance

$$C_X(x, y) := \mathbb{E}X(x)X(y)$$

is related to $\mathrm{cov}_{X^0}(x, y)$ via

$$C_X(x, y) = \mathrm{cov}_{X^0}(x, y) + S_V^2$$

so that it is a function of the difference of its arguments also. If we restrict the process X to a compact window $W \subset \mathbb{R}^d$

$$X_W(x) = X(x)\mathbb{1}_W(x) = S_\Xi * \kappa(x)\mathbb{1}_W(x), \quad x \in \mathbb{R}^d,$$

we obtain the following equivalent of (2.6):

$$\sqrt{2\pi}^d \mathbb{E}|\tilde{X}_W(\xi)|^2 = \mathscr{F}(\gamma_W \cdot C_X)(\xi), \quad \xi \in \mathbb{R}^d.$$

The arguments leading to (2.9) apply for C_X and X_W in a similar way, so we can deduce

$$(\kappa * \kappa^*) * m(x) = \sqrt{2\pi}^d \frac{\overline{\mathscr{F}}(\mathbb{E}|\tilde{X}_W|^2)(x)}{\gamma_W(x)}, \quad x \in \mathrm{int}W. \tag{2.10}$$

The factor $\sqrt{2\pi}^d$ depends on the definition of the Fourier transformation, so that a different definition might yield a different normalization. If, in particular, the above equation is discretized for efficient calculation, the factor must be suitably adapted to the definition of the discrete Fourier transformation.

Let $\{\kappa_\rho\}_{\rho>0}$ be a family of nonnegative functions of compact support on $\mathbb{R}^d$ satisfying

$$(i) \quad \int_{\mathbb{R}^d} \kappa_\rho(x)\mathrm{d}x = 1, \quad \rho > 0,$$

$$(ii) \quad \kappa_\rho(x) = 0, \quad \|x\| \geq \frac{1}{\rho}.$$

Then we have

$$\lim_{\rho\to\infty} (\kappa_\rho * \kappa_\rho^*) * m(x) = m(x)$$

whenever m is continuous in x.

Hence, if we could discretize (2.10), we could obtain an estimator $\hat{m}$ for the surface correlation function,

$$\hat{m}(x) := c_{\mathscr{F}} \frac{\overline{\mathscr{F}}|\mathscr{F} X_W|^2(x)}{\gamma_W(x)}, \quad x \in \mathrm{int}W, \tag{2.11}$$

with a suitable constant $c_{\mathscr{F}}$.

To obtain a discretization of the equation, we need to define a discretization of the process X_W. Since the application of Fourier methods is well-developped and fast Fourier transformation algorithms are available in the context of image processing, it would be convenient to use representations as discrete grey scale images for the quantities involved, that is, for the discretized surface measure S_Ξ and the kernel function κ.

Let $\mathbb{L}^d = t\mathbb{Z}^d, t > 0$, be a lattice with elementary cell

$$C_t = [0,t) \times \ldots \times [0,t).$$

The constant t is called the lattice parameter. A discrete grey scale image is a mapping f from a subset $\mathcal{D}_f \subset \mathbb{L}^d$ to $\mathbb{R}_+$. For a realization S_{Ξ_0} of S_Ξ within a compact window $W \subset \mathbb{R}^d$, we can formally define a discrete grey scale image, for example via the mapping

$$\begin{aligned} f : W \cap \mathbb{L}^d &\to \mathbb{R}_+ \\ x &\mapsto S_{\Xi_0}(C_t + x), \end{aligned}$$

and thus obtain a discrete representation of S_Ξ. Note that, for a point $x \in W \cap \mathbb{L}^d$, we have

$$f(x) = \int_{\mathbb{R}^d} \mathbb{1}_{C_t+x}(y) S_{\Xi_0}(\mathrm{d}y) = \int_{\mathbb{R}^d} \mathbb{1}_{C_t}(y-x) S_{\Xi_0}(\mathrm{d}y) = S_{\Xi_0} * \mathbb{1}^*_{C_t}(x).$$

Hence, with a suitable normalization, the function $\mathbb{1}^*_{C_t}(x)$ can be interpreted as a kernel function and the function f can also be interpreted as a discretization of $S_\Xi * \kappa$.

In practical applications, however, the above definition is of not much use. Usually, instead of a realization of S_Ξ, a realization Ξ_0 of the underlying random closed set Ξ is given, where the corresponding discrete grey scale image is defined by

$$\begin{aligned} f_{\Xi_0} : W \cap \mathbb{L}^d &\rightarrow \mathbb{R}_+ \\ x &\mapsto \mathbb{1}_{\Xi_0}(x). \end{aligned}$$

Here and in the following, we assume that Ξ is a.s. the closure of its interior. From f_{Ξ_0}, no information is given about the underlying continuous surface $\partial\Xi_0$ of Ξ_0, and the surface measure S_{Ξ_0} cannot be reconstructed from the given data. For that reason, the estimator derived above is of little practical value.

Nevertheless, the following observations might help to find a way of estimating the surface correlation function. In the point process context, a relation similar to (2.8) has been derived in [Koc02]. For a stationary simple point process Ψ with intensity measure $\Lambda = \lambda\nu$, the convolution $\big((\Psi-\Lambda)*\kappa\big)$ defines a stochastic process Y whose covariance function can be expressed using the pair correlation function g of Ψ as

$$\mathrm{cov}_Y(v) = \lambda\kappa * \kappa^*(v) + \lambda^2 \int_{\mathbb{R}^d} \kappa * \kappa^*(h)\big(g(v-h)-1\big)\mathrm{d}h$$

for $v \in \mathbb{R}^d$. This formula serves as the basis for the spectral analysis of g in [Koc02]. Now if we define a RACS Σ via

$$\Sigma := \bigcup_{x_i \in \text{ supp } \Psi} (B_r(0) + x_i),$$

then for r sufficiently small, the covariance function $\mathrm{cov}_{\Sigma,V}$ is related to g via

$$\mathrm{cov}_{\Sigma,V}(v) = \lambda \mathbb{1}_{B_r(0)} * \mathbb{1}^*_{B_r(0)}(v) + \lambda^2 \int_{\mathbb{R}^d} \mathbb{1}_{B_r(0)} * \mathbb{1}^*_{B_r(0)}(h)\big(g(v-h)-1\big)\mathrm{d}h.$$

This is the Laue formula, cf. [OM00], p.150. Hence, both the covariance function of the RACS Σ obtained by dilation of suppΨ and the covariance function of the convolution process $Y = \big((\Psi - \Lambda) * \kappa\big)$ are related to g via basically the same formula.

Now if we consider the convolution of S_Ξ with the function $\mathbb{1}_C$, where $C \in \mathcal{C}'$ contains the origin in its interior, we can easily see that $\big(S_\Xi * \mathbb{1}_C\big)$ is a.s. zero outside of the RACS $\Xi' := \partial\Xi + C$, since $x \notin (\partial\Xi + C)$ a.s. is equivalent to $(\check{C}+x)\cap\partial\Xi = \emptyset$ a.s., which yields $S_\Xi(\check{C}+x) = \big(S_\Xi * \mathbb{1}_C\big)(x) = 0$ a.s. Hence, the support of $\big(S_\Xi * \mathbb{1}_C\big)$ is a.s. contained in Ξ'. Note that Ξ' is indeed a RACS by Satz 1.3.4 in [SW00] and inherits stationarity from Ξ. The second order quantities of $\big(S_\Xi * \mathbb{1}_C\big)$ and Ξ' therefore contain similar information.

Moreover, it is reasonable to assume that the covariance $C_{\Xi',V}$ of Ξ' and the surface correlation function m can be related to each other via an analogon of the Laue formula. Then, an estimator for the surface correlation function might be derived from an estimation of $C_{\Xi',V}$. Indeed, in simulation tests it can be observed that for a Boolean model with spherical grains the estimated normalized covariance $p_{\Xi'}^{-2}C_{\Xi',V}$ corresponds well to the normalized surface correlation function $S_V^{-2}m$, cf. chapter 4. The derivation of a formula relating the quantities in question is beyond the scope of this work, but we will show how the normalized covariance of Ξ' can be estimated.

If we can obtain a discrete representation of the stationary RACS Ξ', we can estimate its covariance function via frequency space using the estimator of [Koc02]. There, a discretized version of the function

$$g_{\Xi'}(x) := \mathbb{1}_W(x)\mathbb{1}_{\Xi'}(x)$$

is considered and the covariance is estimated within a compact window W satisfying $o \in \mathrm{int}W$, via

$$\hat{C}_{\Xi',V}(x) = c_{\mathscr{F}} \frac{\overline{\mathscr{F}}|\mathscr{F} g_{\Xi'}|^2(x)}{\gamma_W(x)}, \quad x \in \mathrm{int}W.$$

The volume fraction $p_{\Xi'}$ of Ξ' can be estimated by a point-count method, cf. [SKM95], p.212. Suppose a realization of Ξ' is given within the window W such that the intersection of W with $\mathbb{L}^d$ contains N points $x_0, \dots, x_{N-1}$. Then an unbiased estimator for the volume fraction is given by

$$\hat{p}_{\Xi'} = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\Xi'}(x_i).$$

Consequently, we can estimate the normalized covariance function via

$$\frac{\hat{C}_{\Xi',V}(x)}{(\hat{p}_{\Xi'})^2} = c_{\mathscr{F}} \frac{\overline{\mathscr{F}}|\mathscr{F} g_{\Xi'}|^2(x)}{(\hat{p}_{\Xi'})^2 \gamma_W(x)}, \quad x \in \mathrm{int} W.$$

If an estimate of the volume fraction is given, the estimation algorithm for the normalized covariance contains the following steps.

1. The energy density spectrum $|\mathscr{F} g_{\Xi'}|^2$ of the discrete function $g_{\Xi'}$ is calculated using FFT. This operation has a complexity of $O(N \log_2 N)$.
2. The inverse Fourier transformation $\overline{\mathscr{F}}|\mathscr{F} g_{\Xi'}|^2$ is computed via inverse FFT. This operation also has a complexity of $O(N \log_2 N)$.
3. The result at the point x_k is divided by $\hat{p}^2_{\Xi'}\gamma_W(x_k)$ for every point $x_k, k = 0, \ldots, N-1$. If $\hat{p}^2_{\Xi'}\gamma_W$ is known, these are N arithmetic operations.

Thus, the covariance estimation algorithm has a complexity of $O(N \log_2 N)$.

Note that, if $\mathbb{1}_W$ is represented as a discrete grey scale image, the function γ_W can be calculated using FFT, since

$$\gamma_W = \mathbb{1}_W * \mathbb{1}_W^* = \overline{\mathscr{F}}|\mathscr{F}\mathbb{1}_W|^2,$$

cf. [Bri95], ch.4.5. This operation can be carried out using FFT with a complexity of $O(N \log_2 N)$. In particular, γ_W can be assumed as given in the estimation process as it does not depend on a particular realization of X_W. The use of FFT proves useful especially in the case of the window W having a shape more complex than that of a cuboid or a sphere. In this case, analytic formulae for γ_W are difficult to obtain and point-by-point computation would be rather time-consuming.

Now to use an estimate of the normalized covariance of $\Xi' + C$ for the estimation of the normalized surface correlation function, it is convenient to choose $C = C_{[-t,t]} := [-t,t] \times \ldots \times [-t,t]$. We can show that for a sufficiently fine resolution of the lattice it is possible to reconstruct the discretization of a realization $\partial\Xi_0 + C_{[-t,t]}$ from the grey scale image f_{Ξ_0}. The discrete grey scale function of $\partial\Xi_0 + C_{[-t,t]}$ is defined similarly as for Ξ_0 via the indicator function $\mathbb{1}_{\partial\Xi_0 + C_{[-t,t]}}(x)$ for $x \in W \cap \mathbb{L}^d$.

Consider a hyperplane E of $\mathbb{R}^d$. The intersection of E with the stationary $\mathcal{S}$-set Ξ a.s. is a RACS Ξ_E on E, which, by stationarity of Ξ, is a.s. the closure of its interior with respect to E. For a realization Ξ_0 of Ξ within a window W, we can consider a hyperplane E orthogonal to any of the diagonals of the elementary cell that has nonempty intersection with the lattice $\mathbb{L}^d$. Thus, we obtain the intersection set $\Xi_{E,0}$.

The diagonals of C_t parallel to E span a grid on E that contains the set of lattice points $W \cap \mathbb{L}^d \cap E$ as a subset. The intersection of any of the grid lines with $\Xi_{E,0}$ is a finite collection of disjoint compact segments. If we choose the lattice parameter t smaller than the smallest interval separating these segments, we can make sure that $\partial\Xi_0$ intersects each such line, and, in particular, that it intersects each grid segment joining two points of the lattice in at most one point. In this way, $\partial\Xi_0$ 'separates' lattice points lying inside and outside Ξ_0 on E.

Then for t sufficiently small, $(C_{[-t,t]} + x) \cap \partial\Xi_0 \neq \emptyset$ for $x \in W \cap \mathbb{L}^d$ is equivalent to the condition that there exists at least one point $y \in (C_{[-t,t]} + x) \cap W \cap \mathbb{L}^d$ such that $\mathbb{1}_{\Xi_0}(y) \neq \mathbb{1}_{\Xi_0}(x)$. Since $(C_{[-t,t]} + x) \cap \partial\Xi_0 \neq \emptyset$ is equivalent to $\mathbb{1}_{\partial\Xi_0 + C_{[-t,t]}}(x) = 1$, this condition is fulfilled if, and only if, x is an element of the discrete representation of $\partial\Xi_0 + C_{[-t,t]}$. Given the binary image f_{Ξ_0}, it is easy to determine the points $x \in W \cap \mathbb{L}^d$ satisfying the above condition. Thus, provided the lattice resolution is fine enough, it is possible to reconstruct the discretization of $\partial\Xi_0 + C_{[-t,t]}$ and to estimate its normalized covariance.

Edge effects in Covariance Estimation

The practical application of Fourier methods involves some difficulties which we will sketch in the following. The use of fast Fourier transformation (FFT) algorithms in the calculation of an estimator for the covariance will lead to an effect of overlap. This effect is an inherent property of the discrete Fourier transformation (DFT) of which FFT is simply a special case. It is due to the assumed periodicity of the function to be transformed, cf. [Bri95], ch. 6.4. We will demonstrate this effect in the one-dimensional case for the DFT.

Let h be a real-valued function that is sampled within the observation window $W = [-\frac{T}{2}, (N-1)T + \frac{T}{2}], T > 0$, and let $x_k = kT, k = 0, \dots, N-1$, be the points of observation.
For $n = 0, \dots, N-1$, we then have

$$\left|\mathrm{DFT}(h)(\frac{n}{NT})\right|^2 = \sum_{k=0}^{N-1}\sum_{j=0}^{N-1} h(kT)h(jT)e^{-\frac{2\pi i(k-j)n}{N}}.$$

If we apply the inverse DFT (iDFT), cf. [Bri95], ch. 6.3, we obtain, for

$s = 0, \ldots, N-1$,

$$\mathrm{iDFT}(\left|\mathrm{DFT}(h)\right|^2)(sT) \;=\; \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} h(kT)h(jT)\sum_{n=0}^{N-1} e^{-\frac{2\pi i(k-j-s)n}{N}}.$$

For the third sum we have

$$\sum_{n=0}^{N-1} e^{-\frac{2\pi i(k-j-s)n}{N}} = \begin{cases} N \text{ if } k-j-s \equiv 0 \bmod N, \\ 0 \text{ otherwise,} \end{cases}$$

so that we can rewrite the above equation as

$$\mathrm{iDFT}(\left|\mathrm{DFT}(h)\right|^2)(sT) \;=\; \sum_{k=0}^{N-1} h(kT)h(((k-s) \bmod N)T),$$

$s = 0, \ldots, N-1$. Now consider $(k-s) \bmod N$. Since k and s can take values $\in \{0, \ldots, N-1\}$, $(k-s)$ varies between $1-N, \ldots, N-1$, so that for the index $(k-s)$ here additional values are allowed for which the corresponding points of observation lie outside the window W. Due to the reduction mod N, these values are replaced by values for points inside W, which corresponds to sampling the function h outside W by assuming a periodic extension of h beyond W.

If we insert the grey scale function f_{Ξ_0} of a realization of a RACS Ξ into the last equation,

$$\mathrm{iDFT}(\left|\mathrm{DFT}(f_{\Xi_0})(\frac{n}{NT})\right|^2) \;=\; \sum_{k=0}^{N-1} f_{\Xi_0}(kT)f_{\Xi_0}(((k-s) \bmod N)T),$$

$s = 0, \ldots, N-1$, the same holds true, of course, for f_{Ξ_0}, so that we will collect too many values in the estimation process. This overlapping effect can be avoided easily, however, by extending the grey scale image to the window $2W$ via the trivial extension f_{2,Ξ_0} of f_{Ξ_0} to the set $2W \cap \mathbb{L}^d$ defined by

$$\begin{aligned} f_{2,\Xi_0} : 2W \cap \mathbb{L}^d &\to \mathbb{R}_+ \\ x &\mapsto \mathbb{1}_{\Xi_0}(x)\mathbb{1}_W(x), \end{aligned}$$

cf. [Bri95], ch. 7.3. Extending the window W to avoid the overlapping effect changes the number of values involved in the estimation procedure to $2^n N$. This does not affect the complexity of the estimation algorithm, however.

Edge effects in Classical Power Spectrum Estimation

The overlapping effect described above is due to the assumed periodicity of the functions in the calculation of the DFT. Further effects that should be mentioned in this context are the effect of discrete sampling and the effect of the restriction of the data to a finite window. The first effect is known as aliasing and can be interpreted as an overlap of frequency bands in spectral space. Aliasing can be improved upon by choice of a sufficiently small sampling interval T.

The second effect is called the spectral leakage effect. It is not peculiar to the DFT but is bound to occur in any calculation of a finite Fourier transform. Spectral leakage is due to the fact that restricting a function to a finite window in the original space corresponds to a multiplication of the function with the characteristic function of the window. In frequency space, this multiplication corresponds to the convolution of the respective transforms, cf. [Bri95], ch. 4.6. Sometimes, however, spectral leakage will not be observed in the DFT.

The methods employed to reduce the leakage effect are closely related to the methods used to improve the properties of spectral estimators of second order quantities in random signal analysis. The spectral quantity corresponding to the energy density spectrum for a stationary random signal is the power density spectrum, or simply power spectrum, which describes the distribution of energy per unit time.

Again, we will give an outline in the one-dimensional context.

Assume that a function h is sampled at N points within the observation window W, where the sampling interval T is small enough so that the effects of aliasing can be neglected. For simplicity of presentation, we choose $N = 2M - 1$ and $W = [-(M - \frac{1}{2})T, (M - \frac{1}{2})T]$ symmetric. This is no restriction since the DFT can be defined for any finite window analogously as above. Let $x_n = nT$, $n = -(M-1), \ldots, M-1$, be the points of observation.

It is convenient and instructive in this context to write $h(n) = h(nT)$, to interpret the sampled function as the restriction $\{h(n)\}_{n=-(M-1)}^{M-1}$ of the infinite-duration sequence $\{h(n)\}_{n\in\mathbb{Z}}$ to the window W, and to examine the relationship between the DFT of $\{h(n)\}_{n=-(M-1)}^{M-1}$ and the true Fourier transform of $\{h(n)\}_{n\in\mathbb{Z}}$. For simplicity, we will use the notation h for the sequence $\{h(n)\}_{n\in\mathbb{Z}}$ and $h_{(2M-1)}$ for $\{h(n)\}_{n=-(M-1)}^{M-1}$.

Restricting an infinite sequence to the window W corresponds to a multiplication of the sequence by the rectangular window sequence

$$w_{(2M-1)}(n) = \begin{cases} 1, & -(M-1) \le n \le M-1, \\ 0, & \text{otherwise}, \end{cases}$$

that is, we can write $h_{(2M-1)}(n) = h(n)w_{(2M-1)}(n)$, $n = -\infty, \ldots, \infty$. The subscript $2M-1$ seems awkward but it simplifies later argumentations.

The Fourier transform of the sequence h is called the time-discrete Fourier transform. It is denoted by a capital H and defined by

$$H(\xi) = \sum_{n=-\infty}^{\infty} h(n)e^{-i\xi n}, \quad \xi \in \mathbb{R}^d,$$

cf. [OS75], p.21, where the coefficients $h(n)$ can be recaptured via the inverse Fourier transformation

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\xi)e^{i\xi n}\mathrm{d}\xi, \quad n \in \mathbb{Z},$$

cf. [OS75], p.22. The spectrum H is continuous and periodic of period 2π.

For a sequence of finite duration, which is nonzero exactly for the values $n = -(M-1), \ldots, M-1$, the DFT can be obtained from the time-discrete Fourier transform by sampling at the discrete frequencies

$$\xi = \frac{2\pi k}{2M-1}, \quad k = -(M-1), \ldots, M-1,$$

at distance $\frac{1}{T}$, cf. [KK89]. Thus, for the finite sequence $h_{(2M-1)}$, we have,

$$\mathrm{DFT}(h_{(2M-1)})(\frac{k}{2M-1}) = H_{(2M-1)}(\frac{2\pi k}{2M-1})$$

for $k = -(M-1), \ldots, M-1$. Now for the time-discrete Fourier transform of $h_{(2M-1)}$, we have

$$H_{(2M-1)}(\xi) = \sum_{n=-\infty}^{\infty} h(n)w_{(2M-1)}(n)e^{-i\xi n},$$

which, by the complex convolution theorem, cf. [OS75], is equal to the frequency space convolution of the transforms of h and $w_{(2M-1)}$ given by

$$H_{(2M-1)}(\xi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)W_{(2M-1)}(\xi-\omega)\mathrm{d}\omega, \quad \xi \in \mathbb{R}^d.$$

The transform $W_{(2M-1)}$ can be calculated to

$$W_{(2M-1)}(\xi) = \frac{\sin((2M-1)\xi/2)}{\sin(\xi/2)}, \quad \xi \in \mathbb{R}^d,$$

cf. [Pri81], p.437. Due to the convolution with $W_{(2M-1)}$, the finite transform $H_{(2M-1)}$ is a smeared version of the true transform H. The value of $H_{(2M-1)}$ at a given frequency ξ contains weighted contributions from other frequencies. This effect is called spectral leakage.

For $k = -(M-1), \dots, M-1$, we can write, returning to $N = 2M-1$ for simplicity,

$$\mathrm{DFT}(h_N)(\frac{k}{N}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) \frac{\sin(N(\frac{2\pi k}{N} - \omega)/2)}{\sin((\frac{2\pi k}{N} - \omega)/2)} \mathrm{d}\omega. \tag{2.12}$$

Now the function $\dfrac{\sin(N\xi/2)}{\sin(\xi/2)}$ is zero for $\xi = \pm\frac{2\pi}{N}, \pm\frac{4\pi}{N}, \dots$, so that the convolution in (2.12) yields the true Fourier transform value $H(\frac{2\pi k}{N})$ if, and only if, $H(\omega)$ is zero for $\omega \neq \pm\frac{2\pi}{N}, \pm\frac{4\pi}{N}, \dots$ This is the case if, and only if, H contains spectral lines only at the frequencies $\pm\frac{2\pi}{N}, \pm\frac{4\pi}{N}, \dots$, that is, if, and only if, the sequence h is periodic of a period such that N is an integer multiple, cf. the argumentation in [KK89], p.208. If this is not the case, then the spectral leakage will be apparent in $\mathrm{DFT}(h)(\frac{k}{N})$.

For a general nonperiodic function, leakage as just described cannot be extinguished altogether, but there are methods to reduce the effect. The transform of the window function $w_{(2M-1)}$ has oscillatory lobes, called side lobes, of rather high amplitudes. In the convolution given by (2.12), large values of $W_{(2M-1)}$ correspond to high weights attached to the additional frequency components and thus to a large error. By choosing, instead of $w_{(2M-1)}$, a window function whose Fourier transform has side lobes of smaller amplitudes, the influence of the leakage from the other frequency components can be reduced.

An example for a suitable function is the Bartlett window $w_{B,(2M-1)}$ defined by

$$w_{B,(2M-1)}(n) = \begin{cases} 1 - \frac{|n|}{M}, & -(M-1) \leq n \leq M-1, \\ 0, & \text{otherwise.} \end{cases}$$

The Fourier transform of $w_{B,(2M-1)}$ is given by

$$W_{B,(2M-1)}(\xi) = \frac{\sin^2(\frac{M\xi}{2})}{M \sin^2(\frac{\xi}{2})}, \quad \xi \in \mathbb{R}^d,$$

cf. [Pri81], p.439. It has considerably smaller side lobe amplitudes than the transform of the rectangular window so that a multiplication by the Bartlett window will yield better results for the DFT. For further suitable window functions, cf. e.g. [KK89].

For a stationary discrete random signal h with zero mean, the power spectrum Pow_h is defined as the Fourier transform of the autocorrelation sequence cov_h of h, that is,

$$\mathrm{Pow}_h(\xi) = \sum_{n=-\infty}^{\infty} \mathrm{cov}_h(n) e^{-i\xi n}, \quad \xi \in \mathbb{R}^d.$$

As an estimator for the power spectrum, the *periodogram* I_N is introduced. It is defined as

$$I_N(\xi) = \sum_{m=-(N-1)}^{N-1} \hat{\mathrm{cov}}_h(m) e^{\frac{-i\xi m}{N}},$$

where

$$\hat{\mathrm{cov}}_h(m) = \frac{1}{N} \sum_{n=0}^{N-1-|m|} h(n)h(n+m), \quad -N < m < N,$$

is an estimator for the autocorrelation sequence cov_h of the signal, cf. [OS75]. This can also be written as

$$I_N(\xi) = \frac{1}{N} \left| H_N(\xi) \right|^2, \quad \xi \in \mathbb{R}^d,$$

that is, the squared Fourier transform of the restricted sequence $h(n)w_{N+}(n)$, where

$$w_{N+}(n) = \begin{cases} 1, & 0 \leq n \leq N-1, \\ 0, & \text{otherwise.} \end{cases}$$

It can be shown that

$$\mathbb{E}(I_N(\xi)) = \sum_{m=-(N-1)}^{N-1} \left(1 - \frac{|m|}{N}\right) \mathrm{cov}_h(m) e^{-i\xi m},$$

cf. [OS75], which, as mentioned above, can be interpreted as a convolution in frequency space, that is,

$$\mathbb{E}(I_N(\xi)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{Pow}_h(\omega) W_{B,(2N-1)}(\xi - \omega) \mathrm{d}\omega.$$

The function $W_{B,(2N-1)}$ arises in this context because the true autocorrelation sequence is weighted with the Bartlett window function by the estimator

$\hat{\text{cov}}_h$, which is itself weighted with the rectangular window $w_{(2N-1)}$ in the calculation of I_N, cf. [OS75].

Note that the Bartlett window $w_{B,(2N-1)}$ can be written as a discrete convolution of two rectangular windows of length N, that is,

$$w_{B,(2N-1)}(n) = \frac{1}{N} w_{N+}(n) * w^*_{N+}(n),$$

cf. [KK89], p.250. Hence, the discrete convolution theorem, cf. [OS75], gives

$$W_{B,(2N-1)}(\xi) = \left|W_{N+}(\xi)\right|^2, \quad \xi \in \mathbb{R}^d.$$

Now when calculating the periodogram via $\frac{1}{N}|H_{N+}(\xi)|^2$, the estimator might be improved by chosing a different window function for the data sequence $h(n)$, $n = 0, \ldots, N-1$. This is done indeed by some practitioners, cf. [P$^+$02], for example. As pointed out by [Bra78], p.384, however, this does not necessarily yield correct results, since $\mathbb{E}(I_N)$ in general is not expressible as a frequency space convolution with the true power spectrum in that case.

The periodogram estimator I_N is usually improved instead by weighting the estimated autocorrelation sequence $\hat{\text{cov}}_h$ with a window sequence of length $< 2N - 1$, which is in analogy to the windowing used to reduce the leakage effect as described above. Such a window in time-space is called a *lag window*. The Fourier transform of the lag window is called a *spectral window*. By convolution, it serves to smooth the periodogram, which results in an improved variance of the estimator. Details can be found in [OS75], for example.

Chapter 3

The Surface Correlation Function

As pointed out earlier, the question for the existence of the density of the second moment measure $M^{(2)}_{\Xi,S}$ and the definition of second order quantities related to the surface of a RACS Ξ have not received much consideration in the stochastic geometric literature so far. Since the surface correlation function is an important quantity in trapping and flow problems in the context of random media, however, it has been calculated explicitly for some particle models. Most of these can be interpreted as random closed sets with values in the extended convex ring.
Following a general approach by Torquato as outlined in [Tor02], it is possible to give a representation of the surface correlation function in the context of stochastic geometry. This representation gives rise to an important conjecture concerning a representation of the second moment measure, which will be formulated in the second section.

3.1 The Boolean Model with Spherical Grains

Let Ξ be a stationary isotropic Boolean model with spherical grains of radius R in $\mathbb{R}^d$ and let λ be the intensity of the underlying Poisson point process. The mean volume of the typical primary grain Ξ_0 of Ξ is given by

$$\mathbb{E}\nu(\Xi_0) = R^d \kappa_d$$

so that the volume fraction becomes

$$p = 1 - \exp(-\lambda R^d \kappa_d).$$

Moreover, the specific surface is given by

$$S_V = \lambda(1-p)dR^{d-1}\kappa_d,$$

which follows from eq.(3.2.11) of [SKM95], p.76.

By Satz 1.2.3 in [SW00], the mapping

$$\begin{aligned} \mathcal{F}\times\mathcal{C} &\rightarrow \mathcal{F} \\ (F,C) &\mapsto F+C \end{aligned}$$

is measurable, so that for $\epsilon_1, \epsilon_2 > 0$ via

$$\Xi^{(1)} := \Xi + \epsilon_1 B^d \quad \text{and} \quad \Xi^{(2)} := \Xi + \epsilon_2 B^d$$

two random closed sets are defined. They inherit stationarity and isotropy from Ξ. In addition, $\Xi, \Xi^{(1)}$ and $\Xi^{(2)}$ are jointly stationary, that is, their joint distributions are translation invariant.

By construction, $\Xi^{(i)}$, $i = 1, 2$, is a Boolean model that is uniquely determined by the intensity λ of the underlying Poisson point process and the distribution of the typical primary grain. Since $\Xi^{(i)}$ is obtained from Ξ via dilation by $\epsilon_i B^d$, the primary grain is a sphere of radius $R + \epsilon_i$. Thus, we can calculate the volume fractions

$$p^{(i)} = \mathbb{E}\mathbb{1}_{\Xi^{(i)}}(0) = \mathbb{P}(o \in \Xi^{(i)}) = 1 - \exp(-\lambda(R+\epsilon_i)^d\kappa_d), \quad i = 1, 2.$$

By joint stationarity, the function

$$\mathrm{cov}_V^{(12)}(x,y) := \mathbb{E}\mathbb{1}_{\Xi^{(1)}}(x)\mathbb{1}_{\Xi^{(2)}}(y) - \mathbb{E}\mathbb{1}_{\Xi^{(1)}}(x)\mathbb{E}\mathbb{1}_{\Xi^{(2)}}(y), \quad x, y \in \mathbb{R}^d,$$

is a function of $h = x - y$ and we can write

$$\mathrm{cov}_{V,0}^{(12)}(h) = \mathbb{E}\mathbb{1}_{\Xi^{(1)}}(o)\mathbb{1}_{\Xi^{(2)}}(h) - p^{(1)}p^{(2)}, \quad h \in \mathbb{R}^d.$$

The function

$$C_V^{(12)}(h) := \mathrm{cov}_{V,0}^{(12)}(h) + p^{(1)}p^{(2)} = \mathbb{P}(o \in \Xi^{(1)}, h \in \Xi^{(2)}), \quad h \in \mathbb{R}^d,$$

is called the *cross-covariance* of $\Xi^{(1)}$ and $\Xi^{(2)}$. Note that

$$C_V^{(12)}(h) = \mathbb{P}(o \in \Xi^{(1)}, h \in \Xi^{(2)}) = \mathbb{P}(o \in \Xi^{(1)} \cap (\Xi^{(2)} - h)),$$

so that, in analogy with the covariance C_V of Ξ, for fixed $h \in \mathbb{R}^d$, $C_V^{(12)}(h)$ can be interpreted as the volume fraction of the random closed set

$$\Xi^{(1)} \cap (\Xi^{(2)} - h).$$

This intersection is not in general a Boolean model but we can obtain the following result.

$$\begin{aligned} C_V^{(12)}(h) &= \mathbb{P}(o \in \Xi^{(1)} \cap (\Xi^{(2)} - h)) \\ &= \mathbb{P}(o \in \Xi^{(1)}) + \mathbb{P}(o \in \Xi^{(2)} - h) - \mathbb{P}(o \in \Xi^{(1)} \cup (\Xi^{(2)} - h)) \\ &= p^{(1)} + p^{(2)} - \mathbb{P}(o \in \Xi^{(1)} \cup (\Xi^{(2)} - h)) \\ &= p^{(1)} + p^{(2)} - 1 + \mathbb{P}(o \notin \Xi^{(1)} \cup (\Xi^{(2)} - h)). \end{aligned}$$

Consider the last probability. We have

$$\mathbb{P}(o \notin \Xi^{(1)} \cup (\Xi^{(2)} - h)) = \mathbb{P}(\Xi \cap (B_{\epsilon_1}(0) \cup B_{\epsilon_2}(h)) = \emptyset),$$

which can be evaluated using the capacity functional T_Ξ of Ξ since

$$B_{\epsilon_1}(0) \cup B_{\epsilon_2}(h)$$

is compact. The capacity functional of a Boolean model Ξ with primary grain Ξ_0 and Poisson parameter λ satisfies

$$T_\Xi(K) = 1 - \exp(-\lambda \mathbb{E}\nu(\check{\Xi}_0 + K))$$

for any compact $K \subset \mathbb{R}^d$, cf. [SKM95], p.65. Thus

$$C_V^{(12)}(h) = p^{(1)} + p^{(2)} - 1 + \exp(-\lambda \mathbb{E}\nu(B_R(0) + (B_{\epsilon_1}(0) \cup B_{\epsilon_2}(h)))).$$

Note that

$$B_R(0) + (B_{\epsilon_1}(0) \cup B_{\epsilon_2}(h)) = (B_R(0) + B_{\epsilon_1}(0)) \cup (B_R(0) + B_{\epsilon_2}(h))$$

is the union of two spheres of radii $R + \epsilon_1$ and $R + \epsilon_2$, respectively, whose centres are separated by h. Since this volume clearly depends on $r = \|h\|$ only, $C_V^{(12)}$ is a function of $\|h\|$, so we can write $C_V^{(12)}(r) = C_V^{(12)}(\|h\|)$.

For simplicity of notation, we will write $R_1 := R + \epsilon_1$ and $R_2 := R + \epsilon_2$ in the following. The volume of the union of two spheres of radii R_1 and R_2 at distance $r = \|h\|$ is given by

$$\nu(B_{R_1}(0) \cup B_{R_2}(h)) = \nu(B_{R_1}(0)) + \nu(B_{R_2}(h)) - \nu(B_{R_1}(0) \cap B_{R_2}(h)),$$

the difficult part of which is the intersection volume. Explicit formulae have been given for $d = 3$ and $d = 2$, however, so that we can obtain the following

results. We can assume $R_1 \leq R_2$ without loss of generality.
For $d = 3$ we have

$$\nu(B_{R_1}(0) \cap B_{R_2}(h)) = \begin{cases} \frac{4}{3}\pi R_1^3, & 0 \leq r \leq R_2 - R_1, \\ \frac{4}{3}\pi f_3(r; R_1, R_2), & R_2 - R_1 \leq r \leq R_2 + R_1, \\ 0, & r \geq R_2 + R_1, \end{cases}$$

where $r = \|h\|$ and

$$f_3(r; R_1, R_2) = \frac{-3(R_2^2 - R_1^2)^2}{16r} + \frac{R_2^3 + R_1^3}{2} - \frac{3r}{8}(R_2^2 + R_1^2) + \frac{r^3}{16},$$

cf. [Tor02], p.125. Similarly, for $d = 2$,

$$\nu(B_{R_1}(0) \cap B_{R_2}(h)) = \begin{cases} \pi R_1^2, & 0 \leq r \leq R_2 - R_1, \\ f_2(r; R_1, R_2), & R_2 - R_1 \leq r \leq R_2 + R_1, \\ 0, & r \geq R_2 + R_1, \end{cases}$$

where

$$f_2(r; R_1, R_2) = R_1^2 \arccos\big(\frac{R_1^2 + r^2 - R_2^2}{2R_1 r}\big) + R_2^2 \arccos\big(\frac{R_2^2 + r^2 - R_1^2}{2R_2 r}\big) - \frac{1}{2}\sqrt{D},$$

and $D = (R_2 + R_1 + r)(R_2 + R_1 - r)(R_2 - R_1 - r)(R_1 - R_2 - r)$, cf. [KB97]. Thus we can obtain explicit expressions for the cross-covariance $C_V^{(12)}$ of $\Xi^{(1)}$ and $\Xi^{(2)}$ in the two- and three-dimensional case.

For fixed r and R, $C_V^{(12)}(r) = C_V^{(12)}(r, \epsilon_1, \epsilon_2)$ and the volume fractions $p^{(i)} = p^{(i)}(\epsilon_i), i = 1, 2,$ are continuous as functions of the parameters ϵ_1 and ϵ_2. It is not difficult to see that as ϵ_1 and ϵ_2 tend to zero, the volume fractions $p^{(i)}(\epsilon_i)$ tend to the volume fraction p of Ξ and the cross-covariance $C_V^{(12)}(r, \epsilon_1, \epsilon_2)$ approaches the covariance $C(r)$ of Ξ.

As a continuous function of ϵ_i, the volume fraction $p^{(i)}(\epsilon_i), i = 1, 2,$ can be differentiated with respect to ϵ_i. Similarly, $C_V^{(12)}(r, \epsilon_1, \epsilon_2)$ can be partially differentiated, and in particular the twofold partial derivative $\frac{\partial}{\partial \epsilon_1}\frac{\partial}{\partial \epsilon_2}C_V^{(12)}(r, \epsilon_1, \epsilon_2)$ exists. Certainly the cases of $\epsilon_1 = 0$ and $\epsilon_2 = 0$ are most interesting.

By explicit calculation of the derivatives of $p^{(i)}(\epsilon_i), i = 1, 2,$ for $\epsilon_i = +0$ we obtain

$$\frac{\partial}{\partial \epsilon_i} p^{(i)}(\epsilon_i)\Big|_{\epsilon_i = +0} = \lambda d R^{d-1} \kappa_d \exp(-\lambda R^d \kappa_d),$$

which is equal to the specific surface S_V of Ξ. This relation is well known, cf., for example, [OM00], p.100.

Explicit results of the partial differentiation of $C_V^{(12)}(r, \epsilon_1, \epsilon_2)$ can be given for the two- and three-dimensional case. For $d = 3$ and $\epsilon_1 = \epsilon_2 = +0$, we obtain

$$\frac{\partial}{\partial \epsilon_1} \frac{\partial}{\partial \epsilon_2} C_V^{(12)}(r, \epsilon_1, \epsilon_2)\Big|_{\substack{\epsilon_1=+0\\ \epsilon_2=+0}} = \begin{cases} 16\lambda^2\pi^2R^4(1-p)^2, & r \geq 2R, \\ \left(16\lambda^2\pi^2R^4(\frac{1}{2}+\frac{r}{4R})^2 + \frac{2\lambda\pi R^2}{r}\right) \times \\ \quad\times(1-2p+C(r)), & r < 2R. \end{cases}$$

This is precisely the surface correlation function of the Boolean model with spherical grains of radius R, cf. [Tor02]. This formula was first given by [Doi76] in the context of flow problems in porous media, which is part of the theory of random heterogeneous materials.

In the two-dimensional case, partial differentiation yields

$$\frac{\partial}{\partial \epsilon_1} \frac{\partial}{\partial \epsilon_2} C_V^{(12)}(r, \epsilon_1, \epsilon_2)\Big|_{\substack{\epsilon_1=+0\\ \epsilon_2=+0}} = \begin{cases} 4\lambda^2\pi^2R^2(1-p)^2, & r \geq 2R, \\ \left(4\lambda^2R^2(\pi - \arccos(\frac{r}{2R}))^2 + \frac{4\lambda R^2}{r\sqrt{4R^2-r^2}}\right) \times \\ \quad\times(1-2p+C(r)), & r < 2R. \end{cases}$$

as a formula for the surface correlation function. This formula has not been given explicitly in the literature before.

Similar expressions for a Boolean model with spherical grains of random radius R with distribution function F_R can be obtained, provided R has finite moments of sufficiently high order. The specific surface can be calculated to

$$\frac{\partial}{\partial \epsilon_i} p^{(i)}(\epsilon_i)\Big|_{\epsilon_i=+0} = \lambda d\mathbb{E}(R^{d-1}) \exp(-\lambda\mathbb{E}(R^d)\kappa_d)$$

and the surface correlation function is given, for $d = 3$, by

$$m(r) = \begin{cases} 16\lambda^2\pi^2\mathbb{E}(R^4)(1-p)^2, \quad r \geq 2R, \\ \mathbb{E}\left(16\lambda^2\pi^2R^4(\frac{1}{2}+\frac{r}{4R})^2 + \frac{2\lambda\pi R^2}{r}\right) \cdot (1-2p+C(r)), \\ \qquad\qquad\qquad\qquad\qquad\qquad r < 2R, \end{cases}$$

cf. [Tor02].

The calculations described above are not original, but rather a reformulation of a general derivation by [Tor02] in stochastic geometric terms. In [Tor02], the above formulae are derived via a canonical n-point correlation function and the use of generalized functions. If $\mathbb{1}_\Xi$ denotes the indicator function of the random closed set Ξ, the indicator function of the surface $\partial\Xi$ of Ξ is defined by

$$\mathbb{1}_{\partial\Xi}(x) = |\nabla \mathbb{1}_\Xi(x)|, \quad x \in \mathbb{R}^d,$$

which is a generalized function that is zero except when x is on the surface. For a sphere $B_R(y)$ of radius R centred at an arbitrary point $y \in \mathbb{R}^d$, the indicator function of the surface can be expressed as a simple derivative of $\mathbb{1}_{B_R(y)}$, that is,

$$\mathbb{1}_{\partial B_R(y)}(x) = -\frac{\partial}{\partial R}\mathbb{1}_{B_R(y)}(x), x \in \mathbb{R}^d,$$

loc.cit. The probability function describing the positions of n points in the random model, which corresponds to the volume fraction for $n = 1$ and to the covariance for $n = 2$, can be expressed via spherical indicator functions, so that differentiation of the probability function leads to functions involving surface indicator functions. The same is true for the dilated versions of the random closed set, where differentiation is performed with respect to ϵ_1 and ϵ_2. Of the resulting differentiated functions, the specific surface and the surface correlation function arise as special cases for $n = 1$ and $n = 2$.

A slightly different approach, which allows the extension of the above results to more general classes of random sets, yields the following observations. Remember that for a RACS Ξ on $\mathcal{S}$, not necessarily stationary, the dilated set $\Xi + \epsilon_i B^d$, $i = 1, 2$, is also a RACS. For $x, y \in \mathbb{R}^d$, the mean value functions $p^{(i)}(x)$, $i = 1, 2$, can be defined as usual via the indicator functions of the RACS $\Xi^{(i)}$, $i = 1, 2$, and the cross-covariance $C_V^{(12)}(x, y)$ can be defined similarly as for the stationary Boolean model above.

The derivatives at $\epsilon_i = +0$, $i = 1, 2$, can be expressed as limits, so that for $p^{(i)}(x, \epsilon_i) = \mathbb{E}\mathbb{1}_{\Xi^{(i)}}(x)$, $i = 1, 2$, we can write

$$\begin{aligned}
\frac{\partial}{\partial \epsilon_i} p^{(i)}(x, \epsilon_i)\Big|_{\epsilon_i=+0} &= \lim_{\epsilon_i \to +0} \frac{\mathbb{E}\mathbb{1}_{\Xi+\epsilon_i B^d}(x) - \mathbb{E}\mathbb{1}_{\Xi}(x)}{\epsilon_i} \\
&= \lim_{\epsilon_i \to +0} \frac{\mathbb{E}(\mathbb{1}_{\Xi+\epsilon_i B^d}(x) - \mathbb{1}_{\Xi}(x))}{\epsilon_i} \\
&= \lim_{\epsilon_i \to +0} \frac{\mathbb{E}\mathbb{1}_{(\Xi+\epsilon_i B^d)\setminus\Xi}(x)}{\epsilon_i}.
\end{aligned}$$

Hence, the limit, if it exists a.s., will be proportional to the probability that a point $x \in \mathbb{R}^d$ lies in the ϵ_i-region around Ξ.

Analogously, using the additivity of the expectation, we obtain for the cross-covariance,

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon_1}\frac{\partial}{\partial\epsilon_2}C_V^{(12)}(x,y,\epsilon_1,\epsilon_2)\Big|_{\substack{\epsilon_1=+0\\ \epsilon_2=+0}} \\
&= \lim_{\epsilon_1\to+0}\lim_{\epsilon_2\to+0}\frac{\mathbb{E}(\mathbb{1}_{\Xi+\epsilon_1 B^d}(x)-\mathbb{1}_\Xi(x))(\mathbb{1}_{\Xi+\epsilon_2 B^d}(y)-\mathbb{1}_\Xi(y))}{\epsilon_1\epsilon_2} \\
&= \lim_{\epsilon_1\to+0}\lim_{\epsilon_2\to+0}\frac{\mathbb{E}\mathbb{1}_{(\Xi+\epsilon_1 B^d)\setminus\Xi}(x)\mathbb{1}_{(\Xi+\epsilon_2 B^d)\setminus\Xi}(y)}{\epsilon_1\epsilon_2}, \qquad (3.1)
\end{aligned}$$

which shows that, in case it exists a.s., the limit will be proportional to the probability that the point x lies in the ϵ_1-region around Ξ and the point y lies in the ϵ_2-region around Ξ.

Since the probability that a point x lies on the surface $\partial\Xi$ of Ξ is always zero, the specific surface and the surface correlation function cannot be probability functions as the volume fraction and covariance of Ξ are. But, it is natural to assume such properties as those given above for the specific surface and the surface correlation function. Matheron ([Mat75], p.50) called such a construction for the specific surface the translation of a well-known integral-geometric principle into probabilistic terms. As a definition, the above formulae are not satisfactory since they are not related to the geometric properties of the realization of the random closed set and in general, it is difficult to prove the a.s. existence of functions related to the surface.

However, the above observations give rise to the assumption that if the surface correlation function of a stationary random $\mathcal{S}$-set exists a.s., it will appear as the limit as in (3.1), provided the limit exists a.s.

In conclusion, the a.s. existence of the limit

$$\lim_{\epsilon_1\to+0}\lim_{\epsilon_2\to+0}\frac{\mathbb{E}\mathbb{1}_{(\Xi+\epsilon_1 B^d)\setminus\Xi}(x)\mathbb{1}_{(\Xi+\epsilon_2 B^d)\setminus\Xi}(y)}{\epsilon_1\epsilon_2}$$

can serve as an indicator for the a.s. existence of the surface correlation function for a stationary random closed set Ξ in $\mathcal{S}$. This is the basis for a conjecture concerning a representation of the second moment measure $M^{(2)}_{\Xi,S}$ associated with the surface measure, which is formulated in the next section.

3.2 Limit Representation for the Second Moment Measure $M^{(2)}_{\Xi,S}$

Due to the fact that the support of the surface measure of a stationary random $\mathcal{S}$-set is lower-dimensional and the RACS can have a rather complex surface, it is difficult to gain explicit information on the structure of S_Ξ and the corresponding first and second moment measures. Lebesgue measure is easier to analyze and from the point of view of applications, volume measurements are a fundemental tool and rather easy to perform. Hence, it is tempting to find a representation of S_Ξ and its moment measures in terms of Lebesgue measure. We will sketch how this can be done for the expectation measure $M_{\Xi,S}$ and formulate a conjecture for the second moment measure $M^{(2)}_{\Xi,S}$ as a generalization of this result.

In the following, let Ξ be a random closed set, not necessarily stationary, with values a.s. in the extended convex ring $\mathcal{S}$. Assume that Ξ satisfies the integrability condition (2.1). Furthermore, let S_Ξ, $M_{\Xi,S}$ and $M^{(2)}_{\Xi,S}$ denote the surface measure and corresponding first and second moment measures associated with Ξ, respectively. Remember that the surface measure is defined as $S_\Xi(\cdot) = 2\Phi_{d-1}(\Xi, \cdot)$ where $\Phi_{d-1}(\Xi, \cdot)$ denotes the $(d-1)$-th curvature measure.

In the deterministic context, the first idea to establish a relationship between Φ_{d-1} and Lebesgue measure is the use of the Steiner formula

$$\mu_\epsilon(K, A) = \sum_{j=0}^{d} \epsilon^{d-j} \kappa_{d-j} \Phi_j(K, A),$$

where $\mu_\epsilon(K, A)$ is the volume of the local parallel set of an element K of the convex ring with respect to a bounded Borel set $A \in \mathscr{B}(\mathbb{R}^d)$. Since $\Phi_d(K, A) = \nu(K \cap A)$, the Steiner formula immediately yields

$$\lim_{\epsilon \to +0} \frac{\mu_\epsilon(K, A) - \nu(K \cap A)}{\epsilon} = 2\Phi_{d-1}(K, A),$$

so that we obtain the measure $2\Phi_{d-1}(K, A)$ as the limit, or the derivative with respect to ϵ at the point $\epsilon = +0$, of the local parallel volume $\mu_\epsilon(K, A)$. The volume of the local parallel set can be written as a Lebesgue integral using the indicator function $c_\epsilon(K, A, x)$ of the local parallel set introduced by [Sch80]. This function can be interpreted to count the multiplicities with which to include the point x into the parallel volume. However, counting multiplicities is complicated and

$$\mu_\epsilon(K, A) = \int_{\mathbb{R}^d} c_\epsilon(K, A, x) \mathrm{d}x$$

cannot be written as an integral over A so that there is a limit to the amount of information about $\Phi_{d-1}(K, A)$ we can obtain on the basis of the set A.

In the probabilistic context, the situation is different. From Thm.4.1 in [HL00] it follows as a special case that for any bounded $A \in \mathscr{B}(\mathbb{R}^d)$,

$$\lim_{\epsilon \to +0} \int_A \epsilon^{-1} \mathbb{E} \mathbb{1}_{(\Xi + \epsilon B^d) \setminus \Xi}(x) \mathrm{d}x = M_{\Xi,S}(A)$$

and, more generally,

$$\lim_{\epsilon \to +0} \int_{\mathbb{R}^d} f(x) \epsilon^{-1} \mathbb{E} \mathbb{1}_{(\Xi + \epsilon B^d) \setminus \Xi}(x) \mathrm{d}x = \int_{\mathbb{R}^d} f(x) M_{\Xi,S}(\mathrm{d}x)$$

for each continuous function $f : \mathbb{R}^d \to \mathbb{R}$ of compact support. We will briefly sketch the results obtained in [HL00].

For a set $K \in \mathcal{R}$ and $j = 0, \ldots, d-1$, the curvature measures $\Phi_j(K, \cdot)$ arise as special versions of the generalized curvature measures $\Theta_j(K, \cdot)$ defined on $\mathscr{B}(\mathbb{R}^d \times S^{d-1})$. With a different normalization, a different series of generalized curvature measures $C_j(K, \cdot)$ can be obtained from $\Theta_j(K, \cdot)$, that is,

$$d\kappa_{d-j} C_j(K, \cdot) = \binom{d}{j} \Theta_j(K, \cdot), \quad j = 0, \ldots, d-1,$$

for $K \in \mathcal{R}$. These measures can also be defined for $K \in \mathcal{S}$.

Let $B \subset \mathbb{R}^d$ be a strictly convex body containing the origin in its interior. In [HL00], generalized curvature measures $C_j^B(K, \cdot)$, $j = 0, \ldots, d-1$, with respect to the gauge body B are defined for a convex body K and extended to the extended convex ring. The measures $C_j^B(K, \cdot)$, $j = 0, \ldots, d-1$, are defined on $\mathscr{B}(\mathbb{R}^d \times \mathbb{R}^d)$. The underlying space $\mathbb{R}^d$ is interpreted as a Minkowski space, that is, all distance measurements are taken with respect to the gauge function

$$g(B, x) = \min\{r \geq 0 : x \in rB\}$$

for $x \in \mathbb{R}^d$, where the distance of $x \in \mathbb{R}^d$ to a nonempty set $M \subset \mathbb{R}^d$ with respect to B is defined as

$$d_B(M, x) = \min\{g(B, y - x) : y \in M\}.$$

If the gauge body B is centrally symmetric, then the gauge function defines a norm. Clearly, for $B = B^d$, this norm is simply the Euclidean norm, so that the measures $C_j^{B^d}(K, \cdot)$, $j = 0, \ldots, d-1$, are essentially the same as the generalized curvature measures $C_j(K, \cdot)$ and the results obtained in [HL00] follow for the Euclidean context by using B^d as a gauge body. We will state the results for the Euclidean context and omit the superscript B if $B = B^d$. But first we need to introduce some notation.

For a convex body K, there exists, for any $x \in \mathbb{R}^d$, a unique $y \in K$ such that $d_B(K, x) = g(B, y - x)$. This point is denoted by $y =: p_B(K, x)$ and called the Minkowski projection of x onto K with respect to B. Writing

$$u_B(K, x) = \frac{x - p_B(K, x)}{d_B(K, x)} \in \partial \check{B}$$

for $x \notin K$, the Minkowski normal bundle of K with respect to B is defined as

$$N_B(K) = \{(p_B(K, x), u_B(K, x)) : x \in \partial(K + t\check{B})\}$$

for any $t > 0$. To define the Minkowski normal bundle for a set K from the extended convex ring, the set

$$\Pi_B(K, x) = \{y \in K : d_B(K, x) = g(B, y - x)\}, \quad x \in \mathbb{R}^d,$$

is defined. Using this set, the exoskeleton of K with respect to B can be defined by

$$\mathrm{exo}_B(K) = \{x \in \mathbb{R}^d \backslash K : \left|\Pi_B(K, x)\right| \geq 2\}.$$

For any $x \notin (K \cup \mathrm{exo}_B(K))$, $p_B(K, x)$ is defined as the unique point $y \in \partial K$ satisfying $d_B(K, x) = g(B, y - x)$, and $u_B(K, x)$ is defined as for convex K. For $x \in \mathrm{exo}_B(K)\backslash K$, $(p_B(K, x), u_B(K, x))$ is given some arbitrary but fixed value in $\mathbb{R}^d \times \partial\check{B}$. Then the Minkowski normal bundle for the set $K \in \mathcal{S}$ with respect to B is defined by

$$N_B(K) = \{(p_B(K, x), u_B(K, x)) : x \notin (K \cup \mathrm{exo}_B(K))\}.$$

An important step now is the observation that the measures

$$C_j^B(K, \cdot \cap N_B(K))$$

are nonnegative for K in the extended convex ring, cf. [HL00].

Based on an idea by Matheron ([Mat75]), Schneider ([Sch80]) constructed nonnegative extensions $\overline{\Theta}_j(K, \cdot)$ of the generalized curvature measures $\Theta_j(K, \cdot)$ to the extended convex ring. In [HL00], a similar construction is performed and it is shown that the nonnegative extensions $\overline{C}_j^B(K, \cdot)$ and the measures $C_j^{B,+}(K, \cdot)$ defined by

$$C_j^{B,+}(K, \cdot) := C_j^B(K, \cdot \cap N_B(K))$$

coincide for $j = 0, \dots, d-1$. Furthermore, the identity

$$C^{B,+}_{d-1}(K, \cdot) = C^B_{d-1}(K, \cdot)$$

for any $K \in \mathcal{S}$ is proved and a local Steiner type formula is derived.

For a random closed set Σ with values a.s. in the extended convex ring, $C^+_j(\Sigma, \cdot)$ defines a random measure with intensity measure

$$\Lambda^+_j(\cdot) := \mathbb{E} C^+_j(\Sigma, \cdot), \quad j = 0, \dots, d-1.$$

If the measures $\Lambda^+_j(\cdot \times \mathbb{R}^d)$ are locally finite, the Steiner type formula can be used to establish the vague convergence of the measure $\epsilon^{-1}\mathbb{E}\mathbb{1}_{(\Xi+\epsilon B^d)\setminus\Xi}\nu(\mathrm{d}x)$ to the measure $2\Lambda^+_{d-1}(\mathrm{d}x \times \mathbb{R}^d)$. Further details and the proof, performed in the general Minkowski space context, can be found in [HL00].

Now consider the random closed set Ξ, which by assumption satisfies the integrability condition (2.1). The measures $C^+_j(\Xi, \cdot \times \mathbb{R}^d)$ are obtained from $C_j(\Xi, (\cdot \times \mathbb{R}^d) \cap N_{B^d}(\Xi))$ so that (2.1) ensures the local finiteness of the intensity measures $\Lambda^+_j(\cdot \times \mathbb{R}^d)$, which can be seen as follows. Let A be a bounded measurable subset of $\mathbb{R}^d$. Then for convex bodies K, K_0 with $A \subset \mathrm{int} K, K \subset K_0$,

$$\begin{aligned}
&\mathbb{E}\Big|C^+_j(\Xi \cap K, A \times \mathbb{R}^d)\Big| \\
&\quad = \mathbb{E}\Big|C_j(\Xi \cap K, (A \times \mathbb{R}^d) \cap N_{B^d}(\Xi))\Big| \\
&\quad = \mathbb{E}\Big|\sum_{v \in S(N(\Xi \cap K))} (-1)^{|v|-1} C_j(K_v, (A \times \mathbb{R}^d) \cap N_{B^d}(\Xi))\Big| \\
&\quad \le \mathbb{E} \sum_{v \in S(N_K(\Xi))} C_j(K_v, (A \times \mathbb{R}^d)) \\
&\quad \le \sup_{\substack{K' \in \mathcal{K} \\ K' \subset K_0}} C_j(K', (A \times \mathbb{R}^d)) \mathbb{E} 2^{N(\Xi \cap K_0)} < \infty.
\end{aligned}$$

Since

$$2\Phi_{d-1}(K, \cdot) = 2C_{d-1}(K, \cdot \times \mathbb{R}^d) = 2C^+_{d-1}(K, \cdot \times \mathbb{R}^d)$$

for any $K \in \mathcal{S}$, we obtain $M_{\Xi,S}(\cdot) = 2\Lambda^+_{d-1}(\cdot \times \mathbb{R}^d)$. Hence,

$$M_{\Xi,S}(A) = \lim_{\epsilon \to +0} \int_A \epsilon^{-1} \mathbb{E}\mathbb{1}_{(\Xi+\epsilon B^d)\setminus\Xi}(x)\mathrm{d}x$$

for any bounded Borel set A.

The stationary Boolean model with spherical grains of radius R is an example of such a random set. In the previous section we obtained the equality

$$\lim_{\epsilon\to+0} \frac{\mathbb{E}\mathbb{1}_{(\Xi+\epsilon B^d)\setminus\Xi}(x)}{\epsilon} = S_V, \quad x\in\mathbb{R}^d,$$

for the specific surface. Hence, in this case

$$\begin{aligned} M_{\Xi,S}(A) &= \int_A S_V \mathrm{d}x \\ &= \int_A \lim_{\epsilon\to+0} \epsilon^{-1}\mathbb{E}\mathbb{1}_{(\Xi+\epsilon B^d)\setminus\Xi}(x)\mathrm{d}x \\ &= \lim_{\epsilon\to+0} \int_A \epsilon^{-1}\mathbb{E}\mathbb{1}_{(\Xi+\epsilon B^d)\setminus\Xi}(x)\mathrm{d}x. \end{aligned}$$

More generally, if $M_{\Xi,S}$ has a density $s(x)$ with respect to Lebesgue measure and if the derivative at $\epsilon = +0$ of $\mathbb{E}\mathbb{1}_{\Xi+\epsilon B^d}(x)$ exists for a.e. $x\in\mathbb{R}^d$ and the function $\epsilon^{-1}\mathbb{E}\mathbb{1}_{\Xi+\epsilon B^d}(x)$ can locally be dominated by a locally integrable function $f(x)$, then we may interchange limit and integral and obtain the identity

$$\frac{\partial}{\partial\epsilon}\mathbb{E}\mathbb{1}_{\Xi+\epsilon B^d}(x)\Big|_{\epsilon=+0} = s(x)$$

for a.e. $x\in\mathbb{R}^d$, cf. [HL00].

In the light of the observations made in the previous section concerning the surface correlation function, it is a fair conjecture to assume a similar representation for the second moment measure. This can be stated in the following way.

Conjecture 3.2.1
Let Ξ be a random closed set with values a.s. in $\mathcal{S}$. Under appropriate conditions on the second moment measures

$$\mathbb{E}C_j^+(\Xi,\cdot\times\mathbb{R}^d)C_j^+(\Xi,\cdot\times\mathbb{R}^d)$$

and the cross-correlation measures

$$\mathbb{E}C_i^+(\Xi,\cdot\times\mathbb{R}^d)C_j^+(\Xi,\cdot\times\mathbb{R}^d), \quad i\neq j,$$

where $i, j = 0, \dots, d-1$,

$$\lim_{\epsilon_1 \to +0} \lim_{\epsilon_2 \to +0} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} f(x,y)(\epsilon_1\epsilon_2)^{-1} \mathbb{E}\mathbb{1}_{(\Xi+\epsilon_1 B^d)\setminus\Xi}(x)\mathbb{1}_{(\Xi+\epsilon_2 B^d)\setminus\Xi}(y)\mathsf{d}x\mathsf{d}y$$

$$= c \int\limits_{\mathbb{R}^{2d}} f(x,y) M^{(2)}_{\Xi,S}(\mathsf{d}x, \mathsf{d}y)$$

for any continuous function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ of compact support.

Here, the constant factor c arises from the normalization of the generalized curvature measures.

As far as the author knows, this conjecture or a proof of it has not been stated in the literature before. Independently from this work, Ballani has formulated and proved it, cf. [Bal05]. The conditions for the interchangeability of limits and integrals are the existence of a density with respect to Lebesgue measure of $M^{(2)}_{\Xi,S}$ and the existence of the derivatives of the cross-covariance $C^{(12)}_V$. Since Ξ has values a.s. in $\mathcal{S}$, it has a representation as the union set of a locally finite point process Ψ on the space $\mathbb{R}^d \times \mathcal{K}$. The existence conditions can be formulated as conditions on Ψ so that the absolute continuity of $M^{(2)}_{\Xi,S}$ depends on the properties of Ψ, *loc.cit.*

Chapter 4

Simulation

In this chapter we will demonstrate the correspondence of the estimated normalized covariance of the RACS $\Xi' := \partial\Xi + C_{[-t,t]}$ and the normalized surface correlation function $S_V^{-2}m$ of Ξ. Since this correspondence has not been rigorously established, we will only give an example for the standard Boolean model with spherical grains.

4.1 Estimation of the Surface Correlation Function

We will consider the two-dimensional case. The grains of a stationary isotropic Boolean model Ξ with discs of constant radius R and Poisson parameter λ are strictly convex and the mark distribution of the model is clearly rotation invariant. Hence, the second moment measure $M^{(2)}_{\Xi,S}$ is absolutely continuous with respect to Lebesgue measure.

The surface correlation function of the Boolean model Ξ is given on page 61. It can be written explicitly as

$$m(r) = \begin{cases} 4\lambda^2\pi^2R^2\exp(-2\lambda\pi R^2), & r \geq 2R, \\ \left(4\lambda^2R^2(\pi - \arccos(\frac{r}{2R}))^2 + \frac{4\lambda R^2}{r\sqrt{4R^2-r^2}}\right) \times & \\ \times \exp(-\lambda(2\pi R^2 - 2R^2\arccos(\frac{r}{2R}) + \frac{r}{2}\sqrt{4R^2-r^2})), & r < 2R. \end{cases}$$

Since $4\lambda^2\pi^2R^2\exp(-2\lambda\pi R^2) = S_V^2$, division by S_V^2 yields the normalized surface correlation function

$$S_V^{-2}m(r) = \begin{cases} 1, & r \geq 2R, \\ \left(\frac{1}{\pi^2}(\pi - \arccos(\frac{r}{2R}))^2 + \frac{1}{\lambda\pi^2 r\sqrt{4R^2-r^2}}\right) \times & \\ \times \exp(-\lambda(-2R^2\arccos(\frac{r}{2R}) + \frac{r}{2}\sqrt{4R^2-r^2})), & r < 2R. \end{cases}$$

The Boolean model is simulated within a window of sidelength one such that its representation as a binary grey-scale image contains 500×500 points. The estimation procedure involves averaging over 100 realizations.

We consider three models with different radii and different volume fractions. The first model shown in Fig.4.1 has a volume fraction of 26.97%. The discrete representation of the dilated boundary for this model is shown in Fig.4.2 and the corresponding estimated normalized covariance can be seen in Fig.4.7. For the second model as shown in Fig.4.3, the volume fraction is 39.5%. The dilated boundary is shown in Fig.4.4 and the corresponding estimated normalized covariance in Fig.4.8. The volume fraction of the model shown in Fig.4.5 is 75.67%. Fig.4.6 shows the dilated boundary and Fig.4.9 the estimated normalized covariance of this model.

For the presentation of the data in all diagrams, the theoretical normalized surface correlation function is shown in red and the estimated normalized covariance of Ξ' in blue. The green lines show the sum and the difference of the estimated covariance and the estimated deviation.

The estimated normalized covariance approximates the theoretical normalized surface correlation function well for the first model, but with increasing volume fraction and radius, the quality of the approximation decreases. The reason for this may be seen in the fact that, due to the low volume fraction in the first model, there is relatively little overlap of the grains, in contrast to the second and third model. If there is much overlap and in particular, if many intersections of grains occur such that the angle of intersection is acute, then the reconstruction of the dilated boundary may be incorrect due to insufficient resolution. This may result in the observable underestimation of the covariance. Note that the jump discontinuities of the normalized surface correlation functions at $r = 2R$ are not resolved correctly.

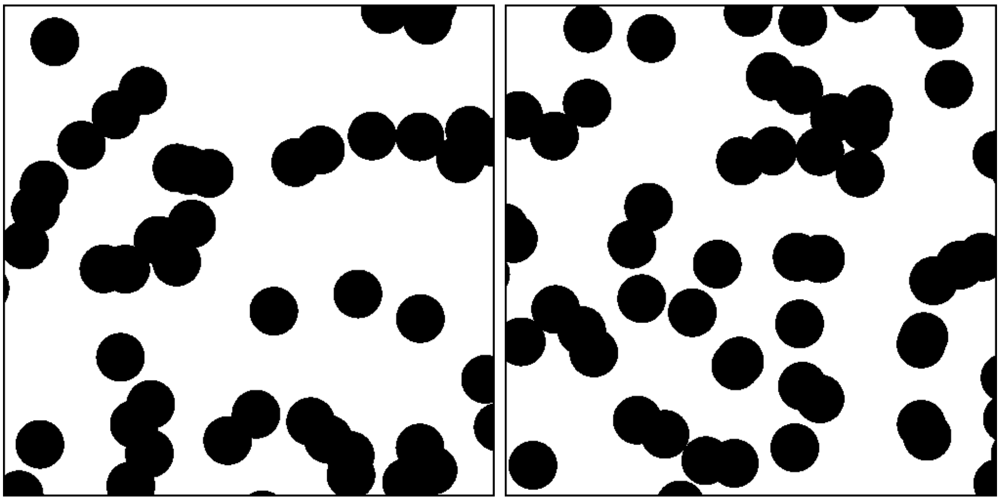

Figure 4.1: Two independent realizations of a Boolean model with discs of constant radius and parameters $R = 0.05$ and $\lambda = 40$

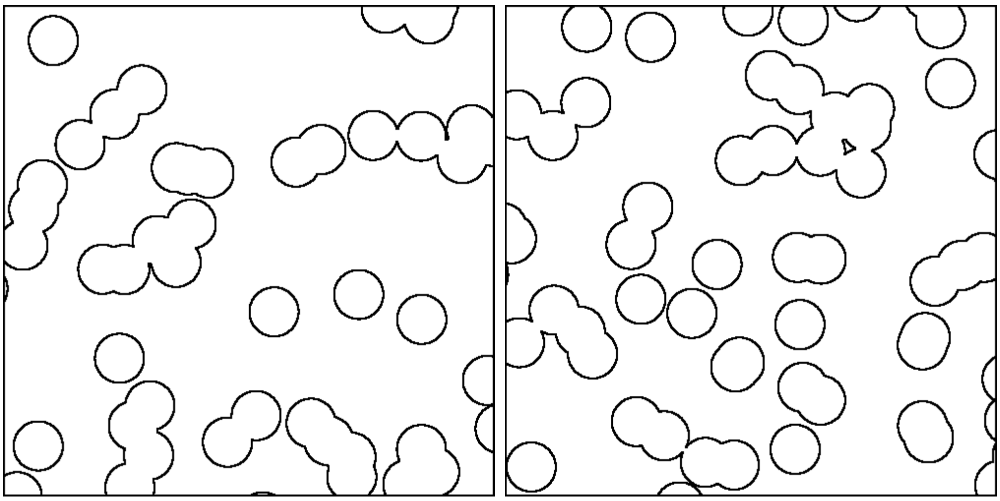

Figure 4.2: Discrete representation of dilated boundary of two independent realizations of a Boolean model with discs of constant radius and parameters $R = 0.05$ and $\lambda = 40$

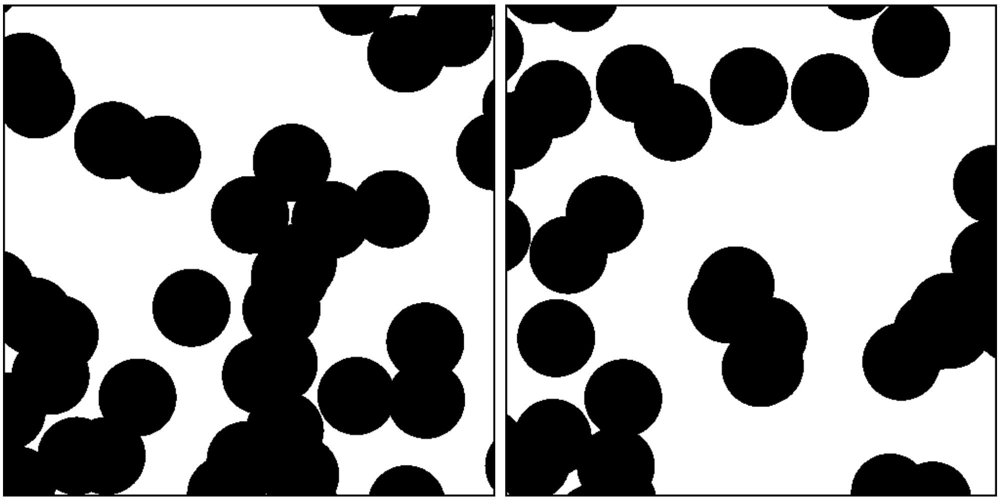

Figure 4.3: Two independent realizations of a Boolean model with discs of constant radius and parameters $R = 0.08$ and $\lambda = 25$

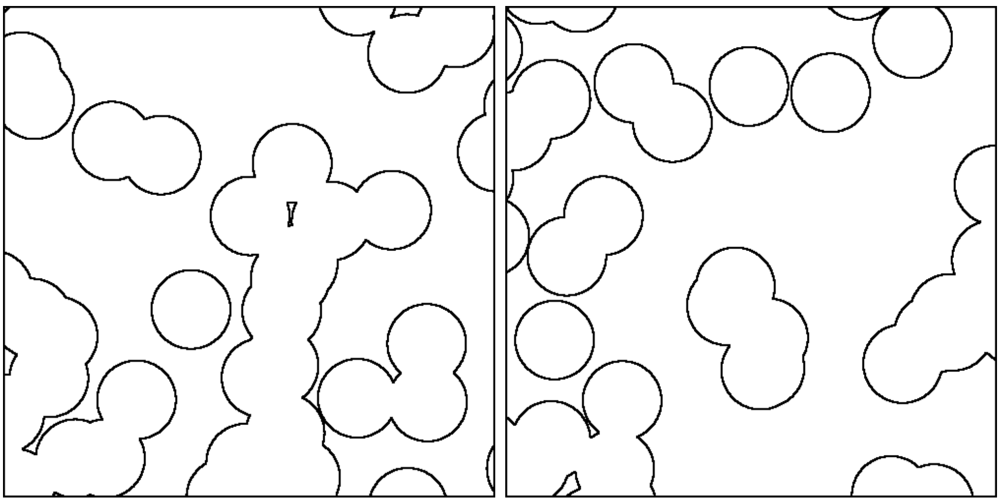

Figure 4.4: Discrete representation of dilated boundary of two independent realizations of a Boolean model with discs of constant radius and parameters $R = 0.08$ and $\lambda = 25$

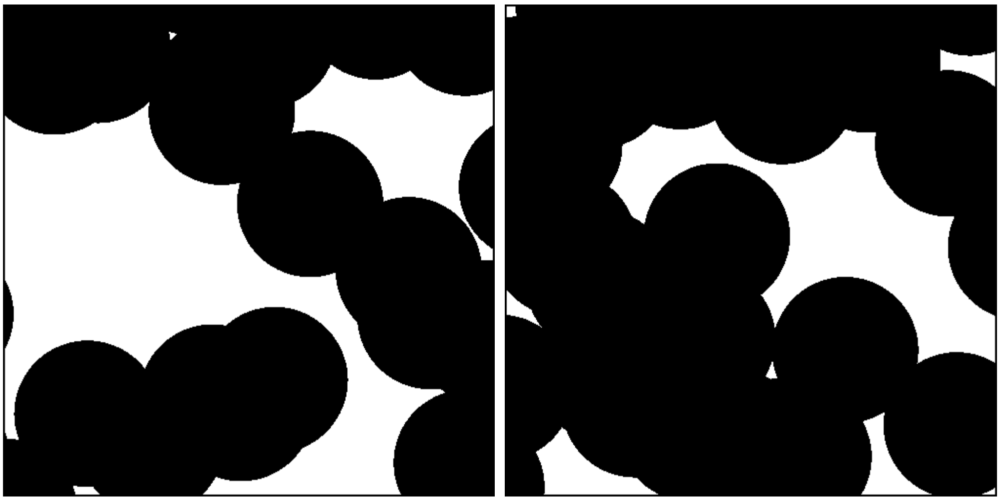

Figure 4.5: Two independent realizations of a Boolean model with discs of constant radius and parameters $R = 0.15$ and $\lambda = 20$

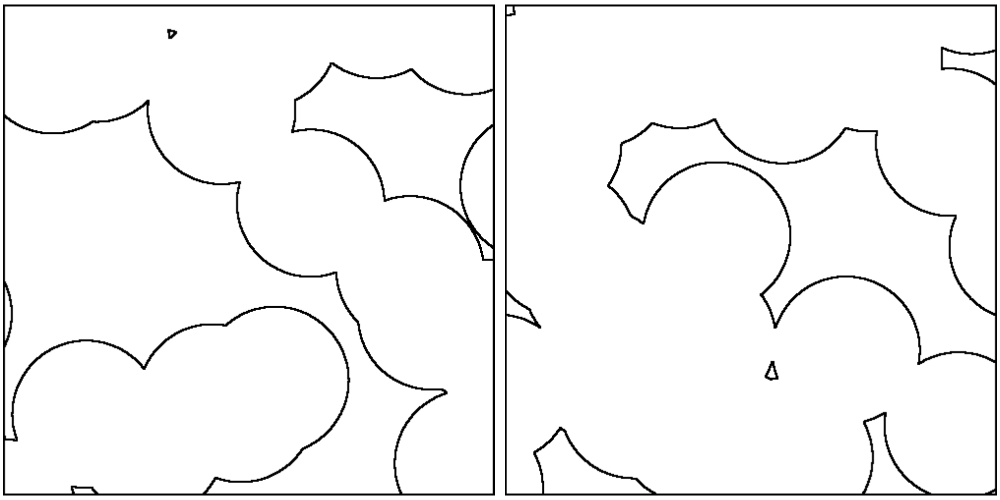

Figure 4.6: Discrete representation of dilated boundary of two independent realizations of a Boolean model with discs of constant radius and parameters $R = 0.15$ and $\lambda = 20$

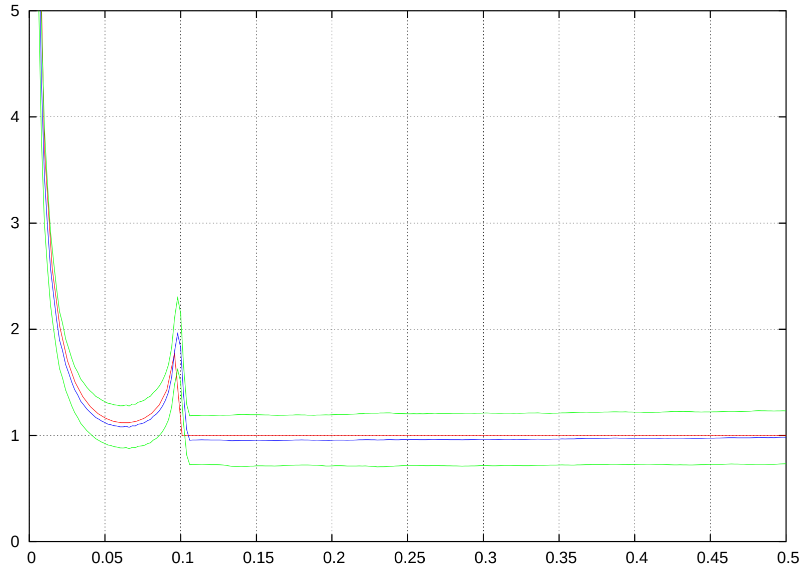

Figure 4.7: Estimated normalized covariance of dilated boundary and normalized surface correlation function of a Boolean model with discs of constant radius and parameters $R = 0.05$ and $\lambda = 40$

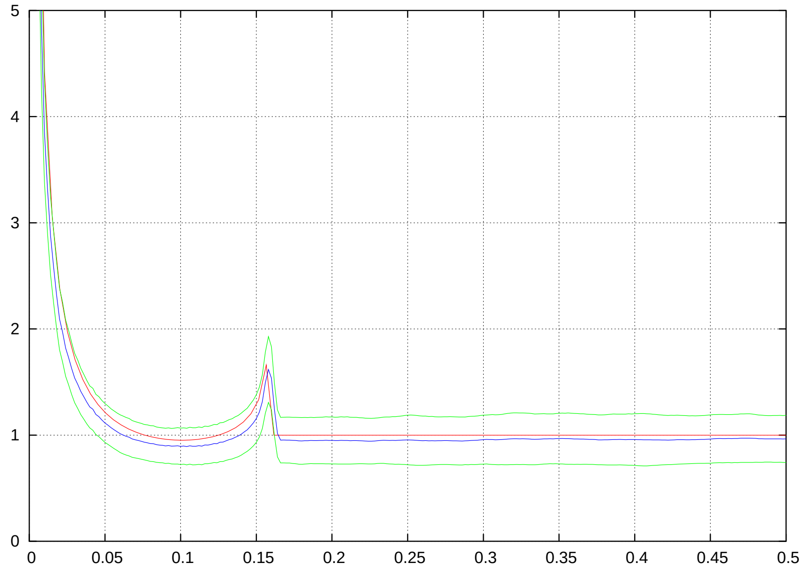

Figure 4.8: Estimated normalized covariance of dilated boundary and normalized surface correlation function of a Boolean model with discs of constant radius and parameters $R = 0.08$ and $\lambda = 25$

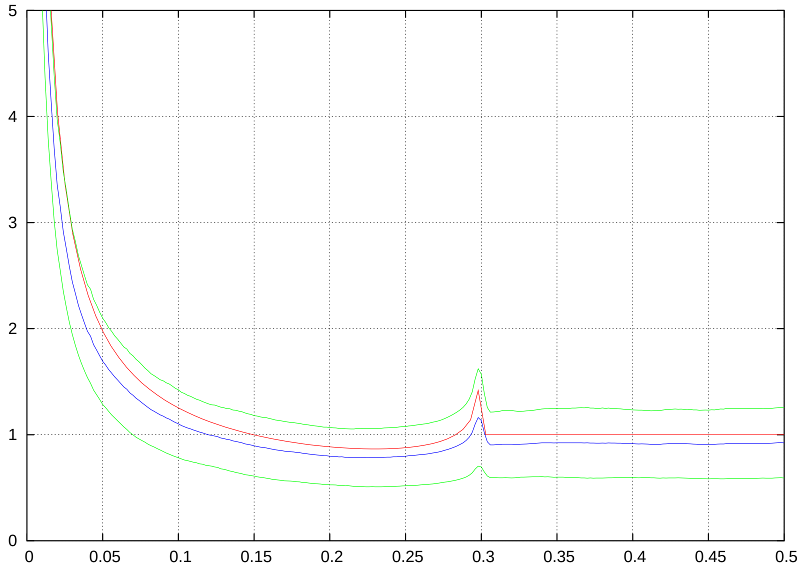

Figure 4.9: Estimated normalized covariance of dilated boundary and normalized surface correlation function of a Boolean model with discs of constant radius and parameters $R = 0.15$ and $\lambda = 20$

Chapter 5

Discussion and Conclusion

To conclude this thesis, we will summarize the results obtained and give an overview of open problems.

First, the existence of a spectral measure associated with the reduced surface covariance measure of a stationary random closed set has been proved. This yields a quantity describing the second order surface structure in frequency space. However, since this quantity is given in many cases as a possibly unbounded measure which is not necessarily absolutely continuous, it is not suitable for analysis.

The proof of an analogon of the theorem of Wiener-Khintchine has served to relate the densities of the reduced second moment and surface covariance measures to spectral quantities that are mathematically more easily tractable. Nevertheless, these quantities are still difficult to use in applications. An idea for the construction of a fast and efficient estimator for the normalized surface correlation function has been given, which remains to be put on a sound mathematical basis. Although the estimator yields reasonable results, a relation between the relevant quantities has still to be established and detailed tests have to be performed.

A derivation of the surface correlation function of a stationary isotropic Boolean model with spherical grains has provided insight into the structure and existence of second order surface quantities. It has served as a basis for a general approach of how to obtain information on the surface correlation function of a random closed set. A conjecture concerning a limit representation of the second moment measure has been deduced.

A further interesting application of spectral methods to the statistical analysis of random closed sets can be found in the second order analysis of curvature measures of lower order. Similar to the surface measure, these lower order curvature measures can be used to define random measures associated with a random closed set. This is of particular interest in materials science. For an open foam, for example, the first curvature measure corresponds, up to a constant factor, to the total edge length. Therefore, the associated second order quantity describes the fluctuation of the edges and thus the filtering properties of the foam. Similarly, the second order structure of the 0th curvature measure characterizes the fluctuation of connectivity of the pore space of autoclaved aerated concrete, which is related to its insulating effect.

Since the material samples are most often analyzed from two- or three-dimensional images, and since fast image processing FFT algorithms are available, estimators for second order quantities based on spectral methods will find good and useful application.

Appendix A

The Covariance Density of a Boolean Model with Spherical Shells

Consider the Boolean model with spherical shells of radius R as in Examples 2.1.1 and 2.1.3. We will derive the formula for the spherical cap cut out of a shell of radius R by the ball of radius r centred at a point on the shell and show the integrability of the covariance density of the model.

Let $x \in \mathbb{R}^d$ be given in polar coordinates, that is,

$$\begin{aligned}
x_1 &= r \sin \varphi_{d-1} \cdots \sin \varphi_3 \sin \varphi_2 \sin \varphi_1 \\
x_2 &= r \sin \varphi_{d-1} \cdots \sin \varphi_3 \sin \varphi_2 \cos \varphi_1 \\
x_3 &= r \sin \varphi_{d-1} \cdots \sin \varphi_3 \cos \varphi_2 \\
&\vdots \\
x_{d-1} &= r \sin \varphi_{d-1} \cos \varphi_{d-2} \\
x_d &= r \cos \varphi_{d-1}
\end{aligned}$$

where $(r, \varphi_1, \varphi_2, \ldots, \varphi_{d-1}) \in [0, \infty) \times [0, 2\pi] \times [0, \pi) \times \ldots \times [0, \pi)$.
The area of the shell $S_R(0) := RS^{d-1}$ is given by

$$R^{d-1}\omega_d = R^{d-1} \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} \prod_{k=2}^{d-1} \sin^{k-1}(\varphi_k) \mathrm{d}\varphi_1 \mathrm{d}\varphi_2 \cdots \mathrm{d}\varphi_{d-1}.$$

A point $x \in \mathbb{R}^d$ lies on the shell $S_R(0)$ if, and only if,

$$\sum_{i=1}^{d} x_i^2 = R^2, \tag{A.1}$$

and within the ball $B_r(y)$ centred at $y \in S_R(0)$ if, and only if,

$$\sum_{i=1}^{d} (x_i - y_i)^2 \leq r^2. \tag{A.2}$$

Hence, a point x lies on the spherical cap cut out of $S_R(0)$ by the ball $B_r(y)$ if, and only if, (A.1) and (A.2) are fulfilled. For simplicity, we can put $y = (0, \dots, 0, R)$ without loss of generality. Then (A.2) becomes

$$\sum_{i=1}^{d-1} x_i^2 + (x_d - R)^2 = \sum_{i=1}^{d-1} x_i^2 + x_d^2 - 2Rx_d + R^2 \leq r^2. \tag{A.3}$$

Solving (A.1) for x_d^2 yields

$$x_d^2 = R^2 - \sum_{i=1}^{d-1} x_i^2,$$

which we can substitute into (A.3) to obtain

$$2R^2 - 2Rx_d \leq r^2.$$

Now if we use polar coordinates to rewrite this inequality as

$$2R^2 - 2R^2 \cos \varphi_{d-1} \leq r^2,$$

we obtain

$$1 - \frac{r^2}{2R^2} \leq \cos \varphi_{d-1}.$$

Since $\cos(\cdot)$ decreases monotonously in $[0, \pi)$, this inequality holds whenever $\varphi_{d-1} \leq \arccos(1 - \frac{r^2}{2R^2})$, which is well-defined only if $r < 2R$. This is no restriction, however, since for $r \geq 2R$, the spherical cap is the shell $S_R(0)$ itself and the area of the spherical cap is given by $R^{d-1}\omega_d$.

For $r < 2R$ we can now calculate the area $F(r)$ of the spherical cap via the integral

$$\begin{aligned} F(r) &= R^{d-1} \int_0^{\arccos(1-\frac{r^2}{2R^2})} \int_0^{\pi} \dots \int_0^{\pi} \int_0^{2\pi} \prod_{k=2}^{d-1} \sin^{k-1}(\varphi_k) \mathrm{d}\varphi_1 \mathrm{d}\varphi_2 \cdots \mathrm{d}\varphi_{d-1} \\ &= R^{d-1}\omega_{d-1} \int_0^{\arccos(1-\frac{r^2}{2R^2})} \sin^{d-2}(\varphi_{d-1}) \mathrm{d}\varphi_{d-1}, \end{aligned}$$

which can be evaluated using the formula

$$\int_{\mathbb{R}} \sin^n(x)\mathrm{d}x = -\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int_{\mathbb{R}} \sin^{n-2}(x)\mathrm{d}x$$

for $n \in \mathbb{N}, n > 0$, cf. [H+96], p.171.

Since for $r = 2R$ the above integral is just $R^{d-1}\omega_d$, the function $F(r)$ is continuous in $r = 2R$ and hence continuous and differentiable for $r \geq 0$. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}F(r) = \begin{cases} \sin^{d-2}(\arccos(1-\frac{r^2}{2R^2}))\frac{1}{\sqrt{R^2-\frac{r^2}{4}}}, & r < 2R \\ 0, & r \geq 2R. \end{cases}$$

By an application of the rule of de l'Hospital, $\frac{\mathrm{d}}{\mathrm{d}r}F(r)$ can be shown to be continuous in $r = 2R$. Hence, the function is continuous for all $r \geq 0$ and, in particular, the reduced covariance density is continuous for all $r \geq 0$ and becomes

$$\mathrm{cov}_S(r) = \begin{cases} \frac{\lambda R^{d-1}}{r^{d-1}}\sin^{d-2}(\arccos(1-\frac{r^2}{2R^2}))\frac{1}{\sqrt{R^2-\frac{r^2}{4}}}, & r < 2R, \\ 0, & r \geq 2R. \end{cases}$$

When interpreted as a function of $x \in \mathbb{R}^d$, cov_S must be modified to

$$\mathrm{cov}_S(x) = \begin{cases} \frac{\lambda R^{d-1}}{\|x\|^{d-1}}\sin^{d-2}(\arccos(1-\frac{\|x\|^2}{2R^2}))\frac{1}{\sqrt{R^2-\frac{\|x\|^2}{4}}}, & \|x\| < 2R, \\ 0, & \|x\| \geq 2R. \end{cases}$$

Since $\mathrm{cov}_S(x) = \mathrm{cov}_S(\|x\|)$ is a continuous radial function of compact support $B_{2R}(0)$, it is integrable over $\mathbb{R}^d$ by Satz 4 in ch. 8.2.II of [Kön00].

Appendix B

Software used

The algorithms used for the estimation of the surface correlation function have been implemented under Linux using existing C++ libraries of the department Models and Algorithms in Image Processing (MAB) of the Fraunhofer Institute for Industrial Mathematics (ITWM) in Kaiserslautern. As the implementation strongly depends on these libraries, the source code is not included in this work.

The Fourier transformation algorithms employed are those provided by the FFTW 3.0.1 library of M. Frigo and S.G. Johnson of Massachusets Institute of Technology (MIT). The FFTW library is freely available for download from http://www.fftw.org.

All diagrams were produced with the aid of gnuplot 3.7, a free interactive data and function plotting utility which can be obtained for free from http://www.gnuplot.info.

The images of the realizations of the Boolean models were visualized by use of MAVI 1.0, an image processing toolbox developped at the MAB department of Fraunhofer ITWM.

Bibliography

[ADD00] R.B. Ash and C.A. Doléans-Dade. *Probability and Measure Theory.* Harcourt, Academic Press, San Diego, Calif., 2nd edition, 2000.

[AGdL74] L. Argabright and J. Gil de Lamadrid. Fourier analysis of unbounded measures on locally compact abelian groups. *Memoirs of the American Mathematical Society*, 145, 1974.

[Bal05] F. Ballani. *Ausgewählte Probleme zur Theorie und Anwendung von zufälligen Keim-Korn-Modellen.* PhD thesis, Technische Universität Bergakademie Freiberg, 2005. to appear.

[Bar63] M.S. Bartlett. The spectral analysis of point processes. *Journal of the Royal Statistical Society B*, 29:264–296, 1963.

[Bar64] M.S. Bartlett. The spectral analysis of two-dimensional point processes. *Biometrika*, 51:299–311, 1964.

[Boc32] S. Bochner. *Vorlesungen über Fouriersche Integrale.* Akademische Verlagsgesellschaft, Leipzig, 1932.

[Bra78] R.N. Bracewell. *The Fourier transform and its applications.* McGraw-Hill, Tokyo, 2nd edition, 1978.

[Bri95] E.O. Brigham. *FFT: Schnelle Fourier-Transformation.* Oldenbourg, München, 1995.

[Coh80] D.L. Cohn. *Measure theory.* Birkhäuser, Boston, 1980.

[Doi76] M. Doi. A new variational approach to the diffusion and the flow problem in porous media. *Journal of the Physical Society of Japan*, 40(2):567–572, 1976.

[DVJ88] D.J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes.* Springer, New York, 1988.

[Fed69] H. Federer. *Geometric Measure Theory.* Springer, Berlin, 1969.

[H+96] W. Hackbusch et al. *Teubner-Taschenbuch der Mathematik.* Teubner, Leipzig, 1996.

[HL00] D. Hug and G. Last. On support measures in Minkowski spaces and contact distributions in stochastic geometry. *The Annals of Probability*, 28(2):796–850, 2000.

[Kal83] O. Kallenberg. *Random Measures.* Akademie-Verlag, Berlin, 1983.

[KB97] P. Krejčíř and V. Beneš. Stereological analysis of spatial surface processes. *Journal of Microscopy*, 186(2):185–197, 1997.

[KK89] K.D. Kammeyer and K. Kroschel. *Digitale Signalverarbeitung.* Teubner, Stuttgart, 1989.

[Koc02] K. Koch. Spektralanalyse zufälliger abgeschlossener Mengen. Master's thesis, Universität Siegen, 2002.

[Kön00] K. Königsberger. *Analysis 2.* Springer, Berlin, 2000.

[KOS03] K. Koch, J. Ohser, and K. Schladitz. Spectral theory for random closed sets and estimating the covariance via frequency space. *Adv. Appl. Prob.*, 35:603–613, 2003.

[Mat75] G. Matheron. *Random sets and integral geometry.* Wiley, New York, 1975.

[MP96] D.G. Manolakis and J.G. Proakis. *Digital Signal Processing.* Prentice Hall, 3rd edition, 1996.

[MR96] M. Mugglestone and E. Renshaw. A practical guide to the spectral analysis of spatial point processes. *Computational Statistics & Data Analysis*, 1996.

[MSSW90] J. Mecke, R. Schneider, D. Stoyan, and W. Weil. *Stochastische Geometrie.* Birkhäuser, Basel, 1990.

[OM00] J. Ohser and F. Mücklich. *Statistical Analysis of Microstructures in Materials Science.* Wiley, Chichester, 2000.

[OS75] A.V. Oppenheim and R.W. Schafer. *Digital Signal Processing.* Prentice-Hall, London, 1975.

[P+02] W.H. Press et al. *Numerical Recipes in C++ : the art of scientific computing*. Cambridge University Press, 2nd edition, 2002.

[Pri81] M.B. Priestley. *Spectral Analysis and Time Series*. Academic Press, New York, 1981.

[RF83] E. Renshaw and E. Ford. The interpretation of process from pattern using two-dimensional spectral analysis: Methods and problems of interpretation. *Applied Statistics*, 32(1):51–63, 1983.

[Sas94] Z. Sasvári. *Positive Definite and Definitizable Functions*. Akademie-Verlag, Berlin, 1994.

[Sch80] R. Schneider. Parallelmengen mit Vielfachheit und Steiner-Formeln. *Geometriae Dedicata*, 9:111–127, 1980.

[Sch93] R. Schneider. *Convex Bodies: the Brunn-Minkowski theory*. Cambridge University Press, 1993.

[SD80] W. Schempp and B. Dreseler. *Einführung in die harmonische Analyse*. Teubner, Stuttgart, 1980.

[SKM95] D. Stoyan, W.S. Kendall, and J. Mecke. *Stochastic geometry and its applications*. Wiley, Chichester, 2nd edition, 1995.

[SW92] R. Schneider and W. Weil. *Integralgeometrie*. Teubner, Stuttgart, 1992.

[SW00] R. Schneider and W. Weil. *Stochastische Geometrie*. Teubner, Stuttgart, 2000.

[Tor02] S. Torquato. *Random Heterogeneous Materials: Microstructure and Macroscopic Properties*. Springer, New York, 2002.